Physics 23 Chapter 9

Rotational Dynamics

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Linear Quantity	Rotational Analog
F	τ
m	I
a	α
F = ma	$\tau = I\alpha$

<u>Example</u>:

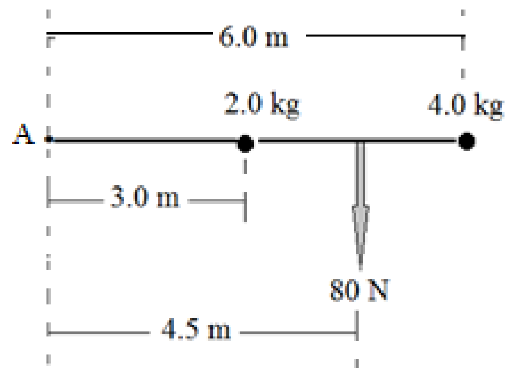
As a wheel whose rotational inertia is 0.40 kg-m^2 sweeps through 48 rads, its angular velocity changes from 12 rad/s to 36 rad/s. What is the total torque acting on the wheel?

$36^2 = 12^2 + 2\alpha\,(48)$
$\alpha = 12$ rad/s^2

$\tau = I\,\alpha$
$= 0.40\,(12)$
$= 4.8$ m-N

Example:

The rod in the figure has a mass of 5.0 kg, and is rotating on a frictionless tabletop about an axis (A) at the left end, perpendicular to the tabletop.



Point masses are at the center and right end of the rod. An 80-N force acting in the plane of the tabletop is applied to the rod, perpendicular to it, 4.5 m from the left end

(a) What is the total torque acting on the rod?

τ = (4.5 m) 80 N
= (360 m-N

(b) What is the rotational inertia of the rod including the point masses? *

$I = (1/3)\ (5.0)\ (6.0)^2 + 2.0\ (3.0)^2 + 4.0\ (6.0)^2$
$= 222$ kg-m^2

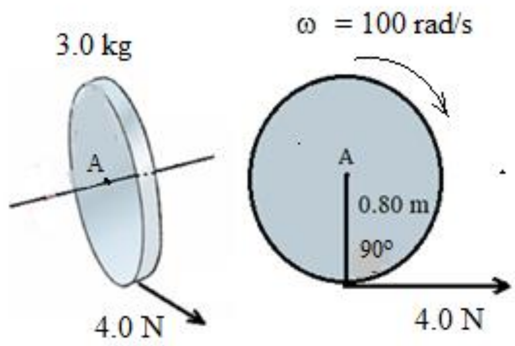
(c) What is the rod's angular acceleration?

$\alpha = \tau / I$
$= 360/222$
$= 1.6$ rad/s^2

*Recall from Chapter 8 that the rotational inertia of a rod whose axis is located at one of its ends is 1/3 ML^2.

Example:

The disk below has a radius R = 0.80 m and a mass M = 3.0 kg. It's initially rotating clockwise at 100 rad/s. A force F = 4.0 N is then applied tangentially to its edge, slowing it.



(a) What is the angular acceleration of the disk?

$I = \frac{1}{2} MR^2$
$= \frac{1}{2} (3.0\ \text{kg})(0.80\ \text{m})^2$
$= 0.96\ \text{kg-m}^2$

$\tau = -LF$ (counter-clockwise)
$= -(0.80\ \text{m})(4.0\ \text{N})$
$= -3.20\ \text{m-N}$

$\alpha = \tau / I$
$= -3.20 / 0.96$
$= -3.33\ \text{rad/s}^2$

(b) At what two times will θ be 800 radians?

$\theta = \omega_o t + \frac{1}{2} \alpha t^2$
$800 = 100\,t + \frac{1}{2}(-3.33)\,t^2$
$t = 9.50$ s
$t = 50.56$ s

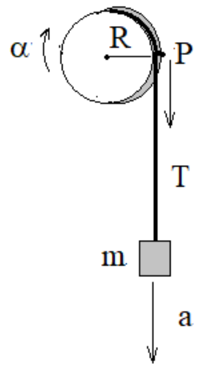
Note: the smaller time above occurs when the disk is still rotating clockwise, while the second time occurs after the disk has come to rest just for an instant, and then began rotating back, counter-clockwise.

Example:

What is the acceleration of the block and the tension in the string in the figure below?

Pulley is rotating clockwise, so the angular acceleration is positive.

$m = 0.70$ kg
$R = 0.20$ m
$I = 0.30$ kg-m^2

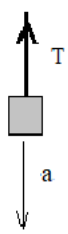


Point P is accelerating downward, so a is negative:

$a = -\alpha R$

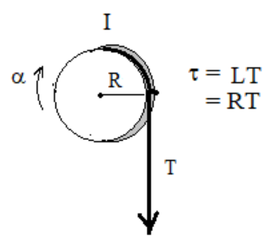
Acceleration of block is the same as the acceleration of Point P.

Apply $F = ma$ to block:



$$F = ma$$
$$T - mg = ma$$
$$T = mg + ma$$
$$T = 6.86 + 0.70\,a \quad (1)$$

Apply $\tau = I\alpha$ to pulley:



$$\tau = I\alpha$$
$$0.20\,T = 0.30\,(-a/0.20)$$
$$= -1.5\,a$$
$$T = -7.5\,a \quad (2)$$

Combining (1) and (2):

$$-7.5\,a = 6.86 + 0.70\,a$$
$$-8.20\,a = 6.86$$
$$a = -0.84 \text{ m/s}^2$$
$$T = -7.5\,(-0.84)$$
$$= 6.27 \text{ N}$$

Angular Momentum vs Linear Momentum

Analogous Quantities

Linear	Angular
m	I
v	ω
$p = mv$	$L = I\omega$
$P = P_o$	$L = L_o$

<u>Example A</u>:

With arms outstretched, an ice-skater has a rotational inertia I_o = 180 kg-m^2, and angular velocity ω_o = 2.0 rad/s. When she pulls in her arms, I = 50 kg-m^2. What then is her angular velocity?

$$L_o = L$$
$$I_o\omega_o = I\omega$$
$$(180)(2) = 50\ \omega$$
$$\omega = 7.2 \text{ rad/s}$$

See video.

<u>Example B</u>:

A star initially making 1.5 rotations per day collapses to one-tenth its initial radius.

Assume the collapsing star remains spherical.

What is the star's final angular speed, in rotations per day?

$$L = L_o$$
$$I\omega = I_o\omega_o$$
$$(2/5)\ M\ (R_o/10)^2\ \omega = (2/5)\ MR_o^2\ (1.5)$$

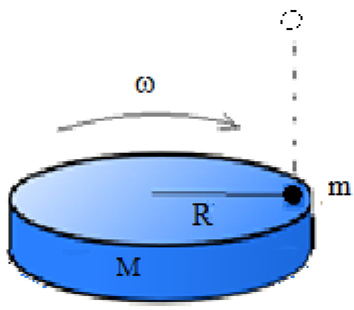
Divide by $(2/5)\ MR_o^2$ to obtain

$$\omega = 150 \text{ rotations/day}$$

Example :

A disk of mass $M = 5.0$ kg and radius $R = 0.50$ m is initially rotating at $\omega_o = 2.0$ rad/s. A point-sized object of mass $m = 1.2$ kg is dropped onto the edge of the disk and sticks to it.

What is the new angular velocity ω of the disk?



BEFORE

$I_o = \frac{1}{2} MR^2$
$\quad = \frac{1}{2} (5.0)(0.50)^2$
$\quad = 0.625$ kg-m^2
$\omega_o = 2.0$ rad/s

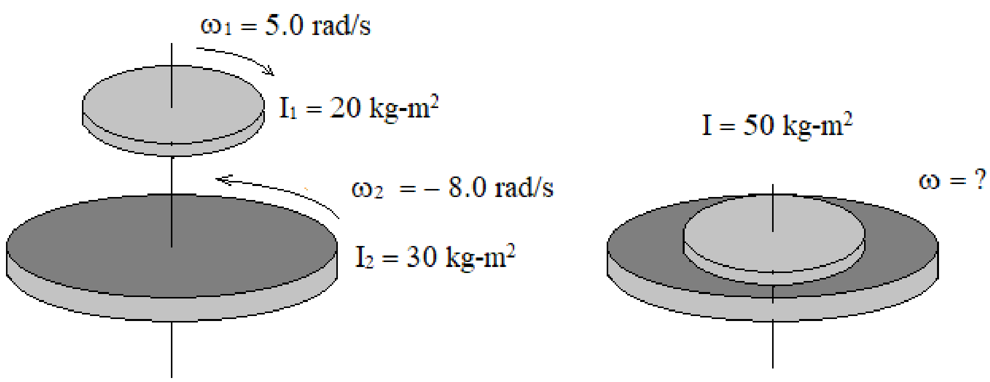
AFTER

The sum of inertias:
$I = \frac{1}{2} MR^2 + mR^2$
$\quad = 0.625 + 1.2 (0.50)^2$
$\quad = 0.925$ kg-m^2

$L = L_o$
$I\omega = I_o\omega_o$
$\omega = (I_o/I)\, \omega_o$
$\quad = (0.625/0.925)\, 2.0$
$\quad = 1.35$ rad/s

Example:

The upper disk below falls onto the lower disk and sticks to it. What is the angular velocity of the combined pair?



L_o = 20 (5.0) + 30 (-8.0)
= - 140 kg-m^2

L = 50 ω

$L = L_o$

50 ω = -140
ω = -2.8 rad/s

Negative angular velocity indicates that the pair of disks is rotating counter-clockwise.

Kinetic Energy of Rotating Objects

Recall the linear kinetic energy equation:

$K = ½ mv^2$

Rotational inertia corresponds to linear inertia (mass), while angular velocity corresponds to linear velocity.

$K = ½ I\omega^2$

Work Done on Rotating Objects

Recall the Linear Case:

If force is constant,
$W = Fx$

Torque τ is the analog of force F, while angular displacement, θ, is the analog of linear displacement, x.

Angular Analog:
If torque is constant,
$W = \tau\theta$

Work-Kinetic Energy Theorem

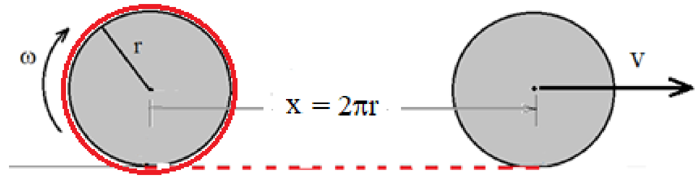
The total work done equals the change in the object's kinetic energy:

$Fx = \frac{1}{2} mv^2 - \frac{1}{2} mv_o^2$

$\tau\theta = \frac{1}{2} I\omega^2 - \frac{1}{2} I\omega_o^2$

Example A:	Example B:
How much work is necessary to angularly accelerate a disk of mass $M = 2.4$ kg and radius $R = 1.6$ m from rest to an angular velocity $\omega = 4.0$ rad/s? Recall: for a disk, $I = ½ MR^2$ $W = \Delta K$ $= K - K_o$ $= K - 0$ $= ½ [½ (2.4)(1.6)^2] (4.0)^2$ $= 24.6$ J	An object subject to a net torque $\tau = 40.0$ m-N experiences a clockwise angular displacement of $\theta = 10$ rotations (20π rad). (a) What total work was done on the object? $W = \tau \theta$ $= 40.0 (20\pi)$ $= 2513$ J (b) What was the change in the object's rotational kinetic energy? $\Delta K = W$ $= 2513$ J

Rolling Motion



The rim of the wheel above is coated with paint. During a certain time t, the wheel completes one rotation (2π radians). Therefore, the wheel's angular velocity during this period of time is

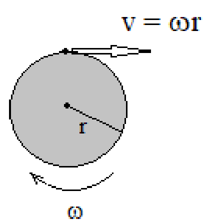
$\omega = \theta/t$
$\quad = 2\pi/t$ (Equation 1)

During the same time, the wheel lays down a circumference-long strip of paint of length $x = 2\pi r$. This distance, x, is the distance traveled by the center of the wheel. Therefore, the linear velocity (also called, "translational velocity) of the center of the wheel is

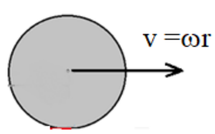
$v = x/t$
$\quad = (2\pi r)/t$

$\quad = (2\pi/t)\, r$
$\quad = \omega r$ (using Equation 1)

The equation $v = \omega r$ looks exactly like the one we use to find the *tangential* velocity of a point on the rim of a rotating wheel. A reminder of that equation is given below.



Tangential Velocity

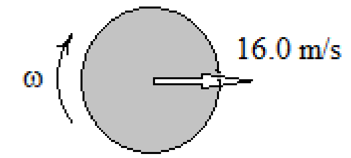


Translational Velocity

The equations for the two velocities are the same, but the two different velocities represent different things.

Example:

What is the angular velocity of a rolling wheel of radius $r = 0.40$ m whose center is traveling at a velocity $v = 16.0$ m/s?



$\omega = v/r$
$= (16.0 \text{ m/s})/0.40 \text{ m}$
$= 40 \text{ s}^{-1}$
$= 40 \text{ rad/s}$

We saw in Chapter 8 that one may suppress the "rad" wherever it might appear if confusion might otherwise arise; we may also *add* the "rad" unit whenever confusion might be caused by its absence, and that's what was done above in reporting the angular velocity as 40 rad/s rather than 40 s^{-1}.

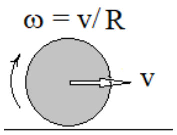
Kinetic Energy of a Rolling Object

Rolling objects have two types of kinetic energy. They have the kinetic energy they would have if they were only moving linearly (sliding) and not rotating, *plus* the energy they would have if they were only rotating and not sliding:

$K = \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2$
$= \frac{1}{2} mv^2 + \frac{1}{2} I (v/R)^2$

Example A:

Obtain an expression in terms of M and v for the total kinetic energy of a ring of mass M rolling at velocity v.



$I = MR^2$

$K = ½\ Mv^2 + ½\ (MR^2)\ (v/R)^2$
$\quad = Mv^2$

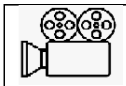
Example B:

What would have to be the velocity of a rolling 5-kg sphere in order that its total kinetic energy be 1400 J?

$(1/2)\ Mv^2 + (1/2)\ (2/5\ MR^2)\ (v/R)^2 = 1400\ J$

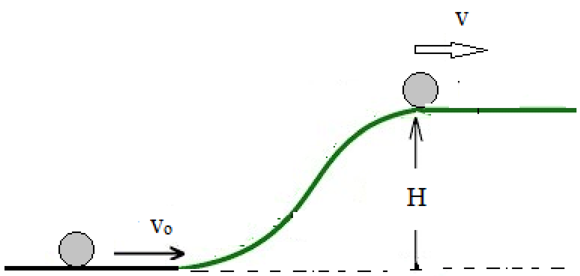
$v = 20.0\ m/s$

Conservation of Energy Involving Rotating and Rolling Objects

	Video Lecture: Sphere Rolling Up Hill This is similar to the *disk* rolling up the hill example below.

<u>Example</u>:

The disk ($I = \frac{1}{2} MR^2$) in the figure rolls at speed v_o up to the top of a hill of height H, arriving there with speed v.



Let the reference elevation be the bottom of the hill and use the conservation of energy law to obtain an expression for v.

$$E = E_o$$

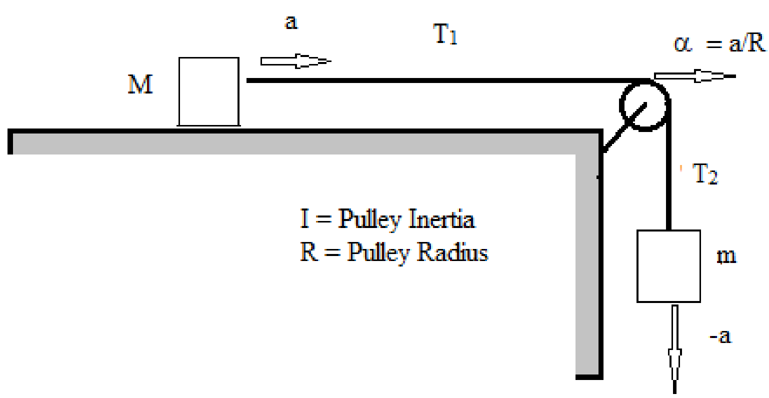
$$\frac{1}{2} Mv^2 + \frac{1}{2} (\frac{1}{2} MR^2)(v/R)^2 + MgH = \frac{1}{2} Mv_o^2 + \frac{1}{2} (\frac{1}{2} MR^2)(v_o/R)^2 + Mg(0)$$

$$v = [v_o^2 - (4/3)gH]^{1/2}$$

	Video Lecture: Pulley Problem Video Lecture: Solution Details

Example:

The block on the frictionless tabletop below is sliding to the right. Obtain an expression for its acceleration in terms of the masses, the rotational inertia, and the pulley radius.



Note that hanging block's acceleration is the negative of the table block's positive acceleration, because the hanging block is accelerating downward, in the negative direction.

There are three unknown quantities in the diagram above. To solve for any one of them, one must create three equations.

Apply Newton's Second Law to the:

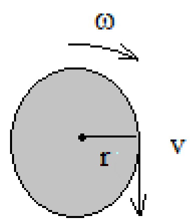
(1) block on the table
(2) hanging block, and
(3) pulley

(1) $T_1 = Ma$
(2) $T_2 - mg = m(-a)$
(3) $RT_2 - RT_1 = I(a/R)$

$$T_2 - T_1 = (I/R^2)\,a$$
$$(mg - ma) - Ma = (I/R^2)\,a$$

$$a = mg/(m + M + I/R^2)$$

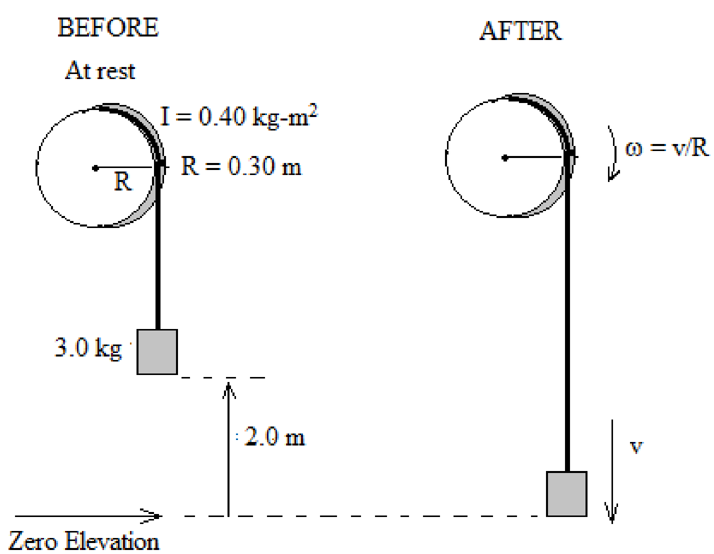
Before studying the example below, recall the following relationship between the angular velocity of a wheel and the tangential velocity of a point on the rim:



$\omega = v/r$

<u>Example</u>:

At the end of a string wrapped around a pulley is an object held at rest. The object is then released and falls. Use the Conservation of Energy Theorem to determine the object's speed when it has fallen 2.0 meters.



$$E_o = E$$

$$\tfrac{1}{2}mv_o^2 + \tfrac{1}{2}I\omega_o^2 + mgh_o = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 + mgh$$

$$0 + 0 + 3.0\,(9.8)\,(2.0) = \tfrac{1}{2}(3.0)\,v^2 + \tfrac{1}{2}(0.40)\,(v/0.30)^2 + 0$$

$$v = 3.97 \text{ m/s}$$

Analogous Quantities: A Review

The table below shows the rotational variables and equations in this chapter that correspond to the linear ones seen in earlier chapters.

Linear	Angular
x	θ
v	ω
a	α
m	I
F	τ
$F = ma$	$\tau = I\alpha$
$K = ½\ mv^2$	$K = ½\ I\omega^2$
$W = Fx$	$W = \tau\ \theta$
$W = \Delta K$	$W = \Delta K$
$p = mv$	$l = I\ \omega$
$P = P_o$	$L = L_o$