Physics 23 Chapter 8
Rotational Motion
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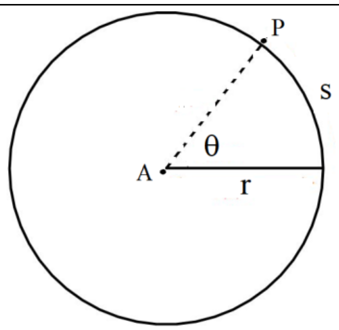
	Video Lecture: Rotational Motion Introduction

The figure below represents a wheel rotating about an axis A at its center. During some period of time the "radial" line sweeps clockwise through an angle θ. Point P at the end of the radial line travels along an arclength s.

Radians



s = Arclength
r = radius

$\theta = s/r$

Units: radians (rad)

Note: θ is the ratio of two distance quantities, and therefore is dimensionless. The rad unit sometimes may be omitted whenever it's convenient. An example of omitting "rad" will occur later.

Examples of degrees-to-radians conversions.

θ (degrees)	s (meters)	θ = s/r (radians)
360	$2\pi r$	2π
180	πr	π
90	$\pi r/2$	$\pi/2$

One radian is approximately 57.3^{o}.

2π rad = 360 degrees
1.0 rad = $360/2\pi$ degrees
= 57.3^{o}

The Angular Motion Equations

The equations that govern rotational motion are identical to the ones governing linear motion, except the linear variable names are replaced with angular ones, as shown below:

The Variables

Angular	**Linear**
θ Angular Displacement Units: radians (rad)	**x** Linear Displacement Units: meters (m)
ω Angular Velocity Units: radians/s (rad/s)	**v** Linear Velocity Units: meters/s (m/s)
α Angular Acceleration Units: rad/s^2	**a** Linear Acceleration Units: (m/s^2)

The Equations

The angular equations shown below are obtained from the linear ones by replacing x with θ, v with ω, and a with α.

	Angular	**Linear**
1	$\theta = \bar{\omega}\, t$	$x = \bar{v}\, t$
2	$\theta = \omega t$ If ω = constant	$x = vt$ If v = constant
3	$\omega = \theta/t$ If ω = constant	$v = x/t$ If v = constant
4	$\omega = \omega_o + \alpha t$	$v = v_o + at$
5	$\bar{\omega} = ½\, (\omega_o + \omega)$	$\bar{v} = ½\, (v_o + v)$
6	$\theta = \omega_o t + ½\, \alpha t^2$	$x = v_o t + ½\, at^2$
7	$\omega^2 = \omega_o^2 + 2\alpha\theta$	$v^2 = v_o^2 + 2ax$

Example A:	Example B:
A wheel initially rotating at 4 rad/s accelerates to 9 rad/s after six seconds. Through how many radians did it rotate during this time? $\theta = \bar{\omega}\, t$ $= ½\, (4 + 9)\, 6$ $= 39$ rad	Solve the problem at the left using $\theta = \omega_o t + ½\, \alpha t^2$ First, find α: $\omega = \omega_o + \alpha t$ $9 = 4 + \alpha\, (6)$ $\alpha = 5/6$ rad/s^2 $\theta = 4\, (6) + ½\, (5/6)\, 6^2$ $= 39$ rad

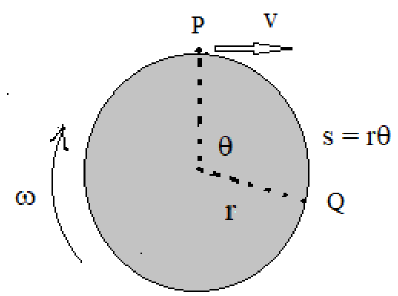
Example A: An object rotating counter-clockwise has an angular velocity of - 6.0 rad/s. It begins accelerating at the rate of 1.8 rad/s^2. After how many seconds will its displacement be 100 rad? $\theta = \omega_o t + \frac{1}{2} \alpha t^2$ $100 = (-6.0)t + \frac{1}{2} (1.8)t^2$ Two solutions: 14.4 s, and -7.7 s. Reject the physically meaningless negative time answer.	Example B: An object is rotating with angular velocity 60 rad/s and slowing down at the rate of -6.0 rad/s^2. Through how many radians will it rotate before it comes to a momentary rest? $\omega^2 = \omega_o^2 + 2\alpha\theta$ $0^2 = 60^2 + 2 (-6.0) \theta$ $\theta = 300$ rad
Example C: Solve the problem in Example B using $\theta = \bar{\omega}$ t. $\bar{\omega} = \frac{1}{2} (\omega_o + \omega)$ $= \frac{1}{2} (60 + 0)$ $= 30$ rad/s Losing 6 rad/s each second will reduce 60 rad/s to zero in 10 seconds: $\theta = \bar{\omega}$ t $= 30 (10)$ $= 300$ rad	Example D: A wheel initially rotating at 6.0 rad/s accelerates to 10.0 rad/s, and during that period of time it makes 10 rotations. What was the wheel's acceleration? $\theta = 10 (2\pi)$ $= 20\pi$ rad $\omega^2 = \omega_o^2 + 2\alpha\theta$ $10.0^2 = 6.0^2 + 2\alpha (20\pi)$ $\alpha = 0.51$ rad/s^2

Tangential and Angular Velocity

<u>Example</u>:

The wheel below is rotating with *constant* angular velocity ω about an axis perpendicular to the wheel and passing through its center.

Points on the rim of the wheel move with constant tangential velocity v.



Calculate the tangential velocity v in terms of ω and r:

In a time t, the radial line sweeps through an angle θ. Consequently,

$\omega = \theta/t$

At the same time, a point on the rim of the wheel sweeps through an arc distance $s = r\,\theta$ from Point P to Point Q. The object's tangential velocity v is given below:

$$\begin{aligned} v &= s/t \\ &= (r\,\theta)/t \\ &= (\theta/t)\,r \\ &= \omega\,r \end{aligned}$$

$$v = \omega r$$

Example : What is the angular velocity of an object located on the surface of Earth? Once every 24 hours every object on Earth completes one rotation of 2π radians (360 degrees). t = 24 hours = 24 hr (3600 s/hr) = 8.64×10^4 s $\omega = \theta/t$ = (2π rad)/ (8.64×10^4 s) = 7.27×10^{-5} rad/s	

Example:

What is the tangential velocity of a point on the surface of Earth at a particular latitude?

Solution:

The radius of Earth is $R = 6.38 \times 10^6$ m.

Points at a certain latitude θ travel in a circular path whose radius r is given by the equation

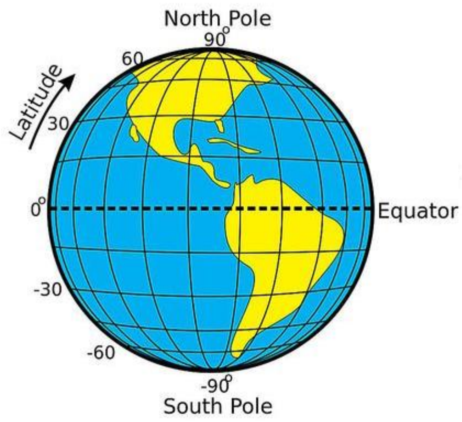
$r = R \cos\theta$

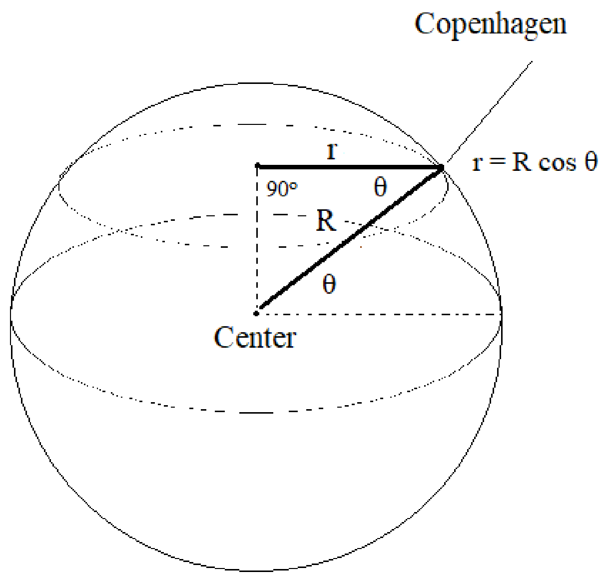
The angular velocity of *any* point on Earth is the value found in the previous example:

$\omega = 7.27 \times 10^{-5}$ rad/s

Tangential Velocity:

$v = \omega r$
$\quad = \omega R \cos\theta$





Example:

What is the tangential velocity of objects in Copenhagen, Denmark?

Copenhagen is at 55^o north latitude.

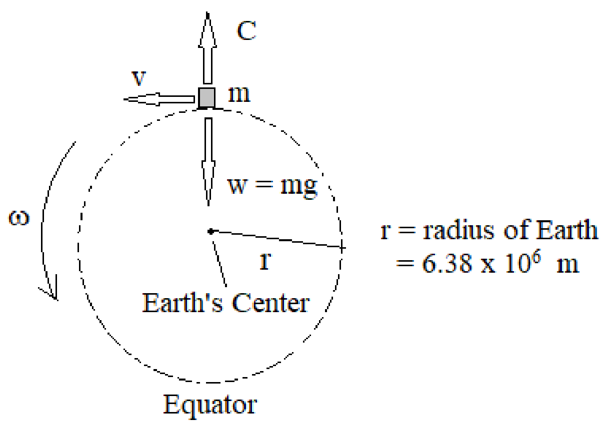
Using the results of the previous example:

$r = R \cos 55^o$
$\quad = (6.38 \times 10^6) \cos 55^o$
$\quad = 3.66 \times 10^6$ m

$v = \omega r$
$\quad = 7.27 \times 10^{-5} (3.66 \times 10^6)$
$\quad = 266$ m/s

Example A:

What would have to be the angular velocity of Earth in order that the apparent weight (C) of a person at the equator be zero?



The positive radial direction above points from the object toward Earth's center, so C_r is negative, while w_r is positive.

$$F_r = ma_r$$
$$w_r + C_r = ma_r$$
$$mg - C = mv^2/r \quad \text{(Set C = 0)}$$
$$mg = mv^2/r$$

$$v = (gr)^{1/2}$$
$$= (9.8 \times 6.38 \times 10^6)^{1/2}$$
$$= 7907 \text{ m/s}$$

$$\omega = v/r$$
$$= 7907 \text{ m/s} / 6.38 \times 10^6 \text{ m}$$
$$= 1.24 \times 10^{-3} \text{ /s}$$
$$= 1.24 \text{ rad/s}$$

Note: The dimensionless "rad" unit may be legally inserted in any expression if doing so makes sense. In this case, reporting the answer in rad/s rather than in reciprocal seconds makes it clearer that the quantity is an angular velocity.

Example B:

Using the angular velocity (1.24×10^{-3} rad/s) obtained in the example at the left, how many rotations would Earth have to make *per day* to have that angular velocity?

In one second, the number of radians swept through would be

1.24×10^{-3} rad

In one day (86,400 seconds), Earth would sweep through

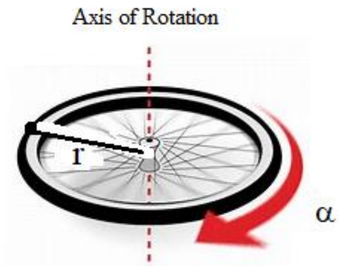
$86{,}400\,(1.24 \times 10^{-3}) = 107$ radians

In one day, Earth would complete

$107/2\pi = 17$ rotations

Thus, in a 24-hour period Earth residents would witness 17 sunsets and 17 sunrises.

Tangential and Angular Acceleration



$a = \alpha r$

Proof:

$$a = \frac{(v - v_o)}{t}$$

$$= \frac{(\omega r - \omega_o r)}{t}$$

$$= \frac{(\omega - \omega_o)}{t} \; r$$

$$= \alpha r$$

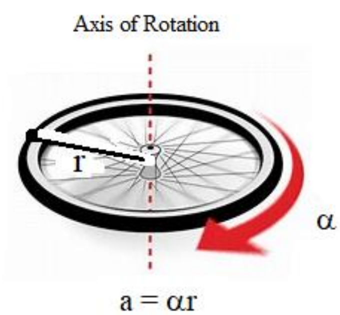
$$\boxed{a = \alpha r}$$

To the list of rotational motion equations provided before, we add two more--the bottom two in the table below:

	Angular Equations
1.	$\theta = \bar{\omega}\, t$
2.	$\theta = \omega t$ If ω = constant
3.	$\omega = \theta / t$ If ω = constant
4.	$\omega = \omega_o + \alpha t$
5.	$\bar{\omega} = \frac{1}{2}(\omega_o + \omega)$
6	$\theta = \omega_o t + \frac{1}{2}\alpha t^2$
7.	$\omega^2 = \omega_o^2 + 2\alpha\theta$
8.	$v = \omega\, r$
9.	$a = \alpha\, r$

Example

The angular acceleration of a wheel is 4.0 rad/s^2. What is the tangential acceleration of a point on the wheel that is 1.4 m from the axis of rotation?



$a = \alpha r$

$= (4.0 \text{ rad/s}^2)(1.4 \text{ m})$

$= 5.6 \text{ rad-m/s}^2$

$= 5.6 \text{ m/s}^2$

Radians are dimensionless, so we may omit the “rad” in order that the proper units of acceleration (m/s^2) appear.

Torque

	Video Lecture: Torque Intro

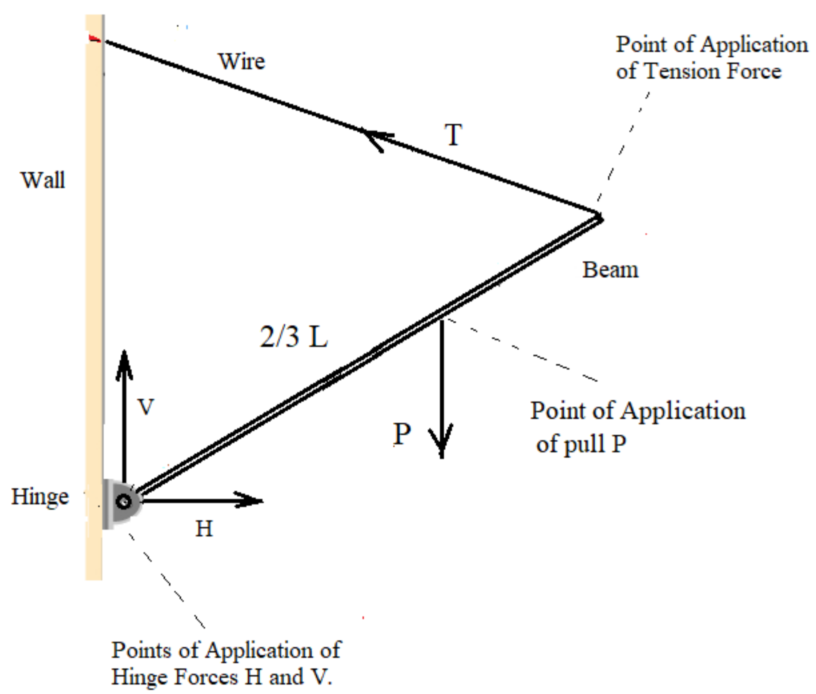
A "torque" is an action that can cause an angular acceleration of an object. Before we learn how to calculate torques, we need to discuss the following:

Points of Application of Forces
Lever Arms
Perpendicular Parts of Forces

Points of Application of Forces

A beam of length L and of negligible weight is hinged to a wall in the figure below. A wire connects the upper end of the beam to the wall.

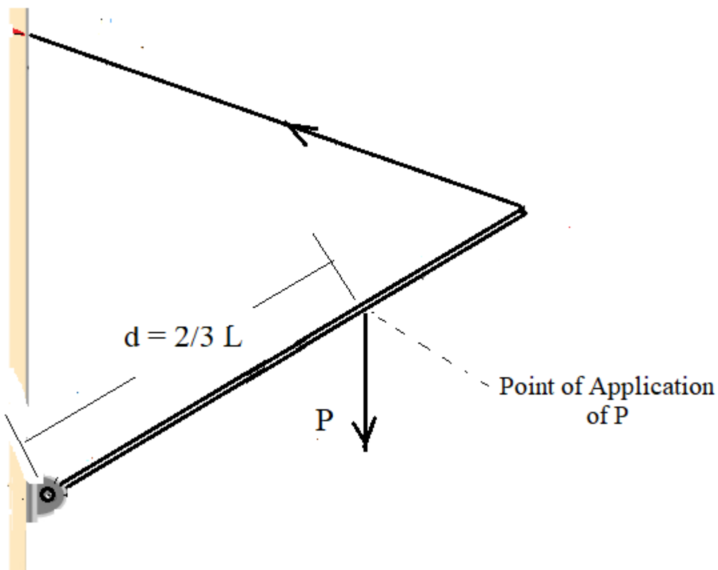
Four forces act on the beam: The horizontal (H) and vertical (V) components of the hinge force, the tension force (T), and a downward pull (P) applied at a point two-thirds of the beam length from the axis. The "points of application" of these forces are indicated. Earth's pull on the beam (whose weight is "negligible") is assumed to be very small in comparison to the other forces acting and so is ignored.



Lever Arms

"Lever arms" associated with forces are distances (d) from the axis of rotation to the point of application of the force.

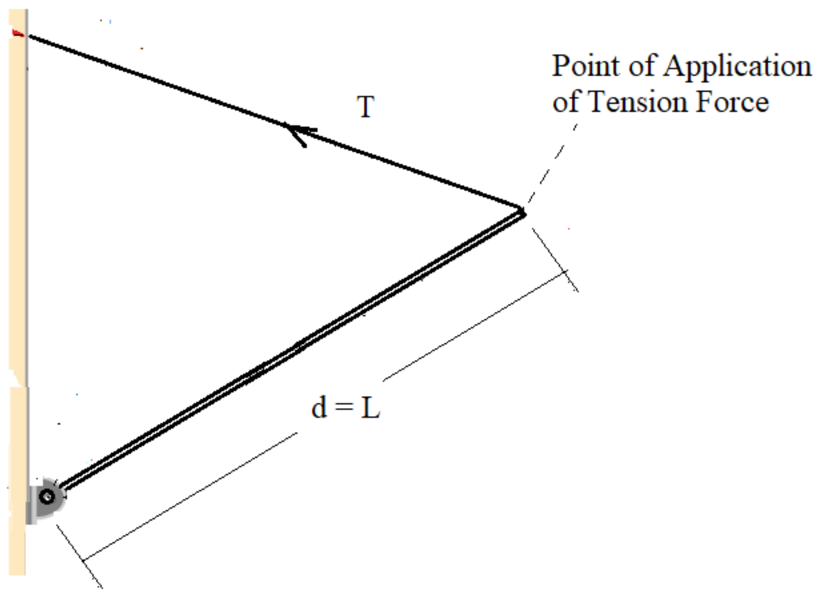
As an example of a lever arm, consider the pull P applied at a distance 2/3 L from the hinge: the lever arm associated with the pull P is
d = 2/3 L



Other lever arms are described below.

Lever Arm of Tension Force T

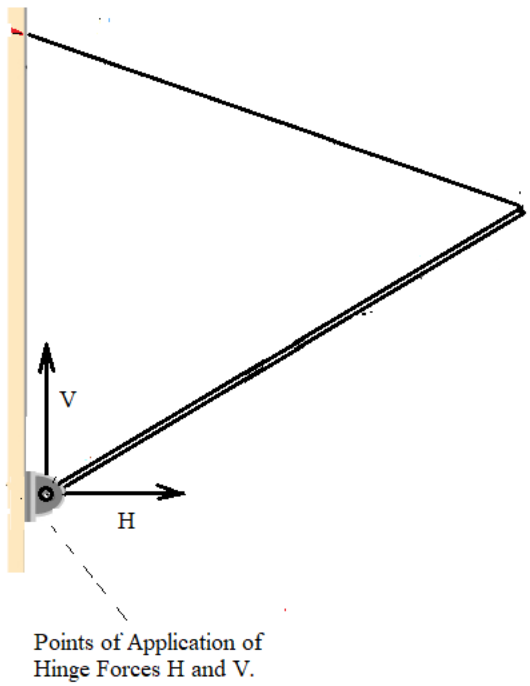
The distance between the point of application of the tension force and the axis (the hinge) is the length of the beam, L.



The lever arm of the tension force is L:

Other forces are not shown.

Lever Arms of Hinge Forces H and V



The points of application of the two hinge forces H and V are located at the axis of rotation. The distance from the axis to the point of application in each case is zero so the lever arms are zero:

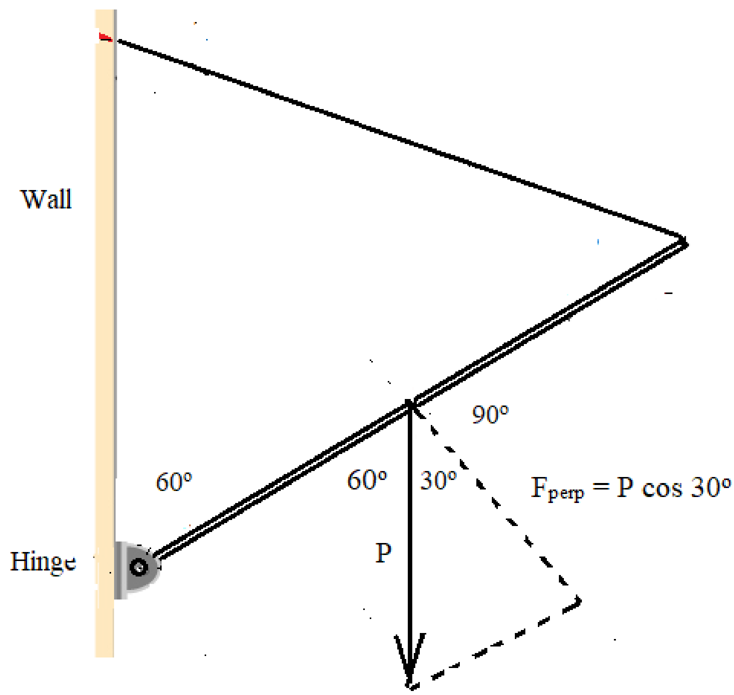
$$d = 0$$

Other forces are not shown.

Perpendicular Parts of Forces

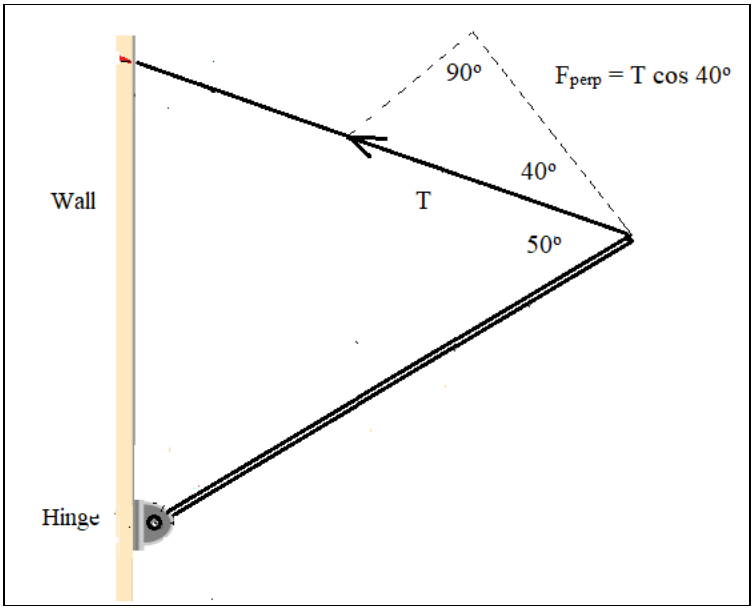
The perpendicular part of a force is the component (part) of the force that is perpendicular to the lever arm.

The perpendicular part of the pull force P force is shown below. Other forces are not shown.



Perpendicular Part of the Tension Force

The perpendicular part of the tension force is shown below.
Other forces are not shown.



Lever arms for H and V hinge forces do not exist, so they have zero perpendicular parts.

Torque

The "torque" of a force is the product of the lever arm (d) and the perpendicular part of the force (F_{perp}):

$$\tau = \pm\, d\, F_{perp}$$

We use the positive (+) sign if the torque would cause the beam to rotate clockwise; the negative sign (-) is used if the torque would cause a counter-clockwise rotation.

Torque Due to Tension

Based on these rules, we see that the torque due to the tension in the wire would cause a counter-clockwise rotation, so the expression for its torque is

$$\tau = -L\,(T \cos 40^o)$$

Torque Due to Pulling Force P

The torque due to the pulling force P is positive:

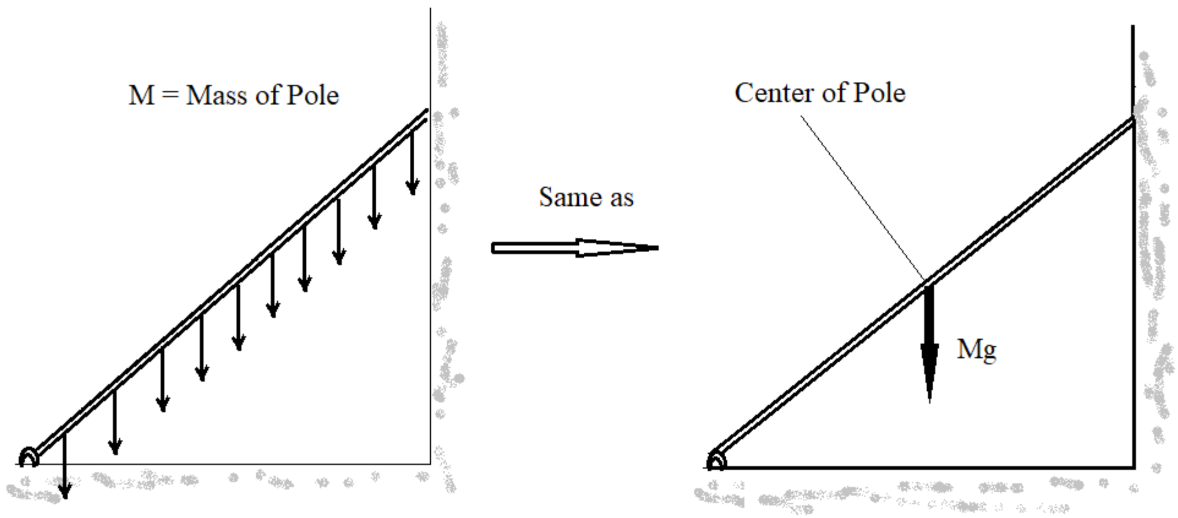
$$\tau = (2/3)L\,(P \cos 30^o)$$

The torques due to the hinges forces V and H are zero because--as we have noted already--their lever arms (d) are zero.

In the discussion above of forces acting on a rotatable object we dealt with the tension force, the pulling force, and the two hinge forces, but we ignored one force: Earth's pull on the object. We ignored that force because we assumed that the weight of the pole was too small compared to the other forces acting to make a difference in our calculation.

What do we do when the object's weight is not ignorably small? The answer is below.

Consider the pole below propped up against a wall. Earth is pulling downward on every one of the nearly infinite bits of matter that make up the pole. Without proof, we declare that to account for the near-infinite number of torques these pulls exert on the pole, we may imagine that *all* of the mass of the pole is concentrated at its center.* The torque due to that imagined single force is the same as the sum of all nearly-infinite number of infinitesimally-small torques.



* Note: treating the rod as if all of its mass is at the center is valid provided the rod's mass is distributed "uniformly," that is, one side is not heavier that the other side. You may assume that the mass in all of the objects we deal with in this chapter is distributed uniformly.

Example:

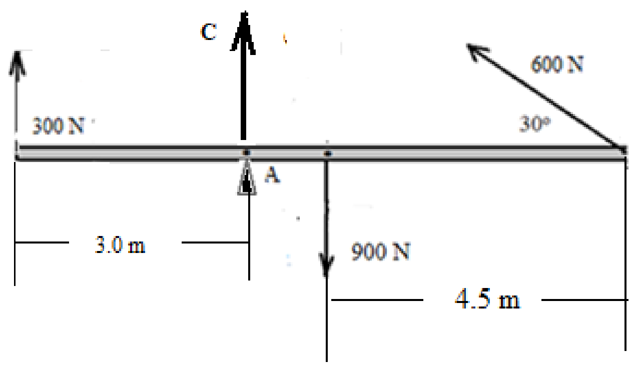
The rod in the figure below is 9.0 meters long and weighs 900 N, and the rod's mass is distributed uniformly.

The rod is resting on a "fulcrum" located at A, 3.0 meters from the left end, which serves as the axis about which the pole might rotate.

At the left end of the rod a 300-N force pushes upward. At the other end of the rod at 600-N force pulls upward and to the left on the rod, at the 30^{o} angle indicated.

The rod is "in equilibrium," meaning that it's not accelerating linearly or angularly.

The effect of Earth's 900-N pull on the uniform rod is the same as if we imagine that all of the mass of the rod is concentrated at its center, 4.5 m from either end.



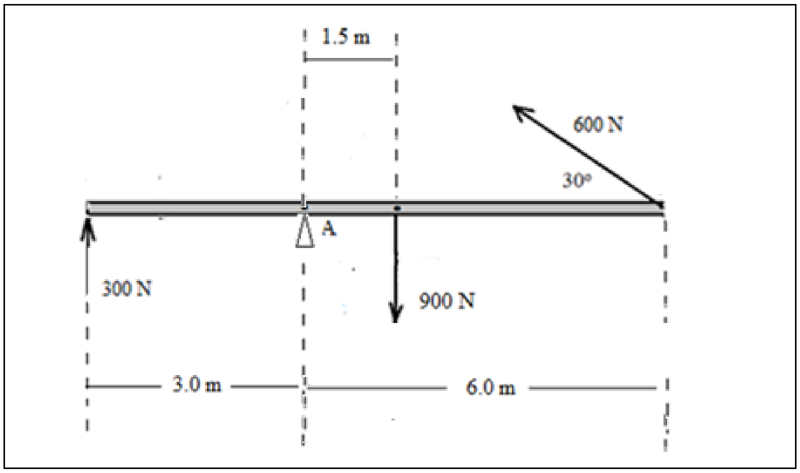
The fulcrum is labeled "A" to indicate the location of the axis about which the rod could possibly rotate. The fulcrum exerts a contact force C on the rod.

Note that the weight of the rod is represented by a single force vector of magnitude 900 N imagined to act at the 9.0-meter rod's center.

Not shown above is the contact force between the fulcrum and the rod.

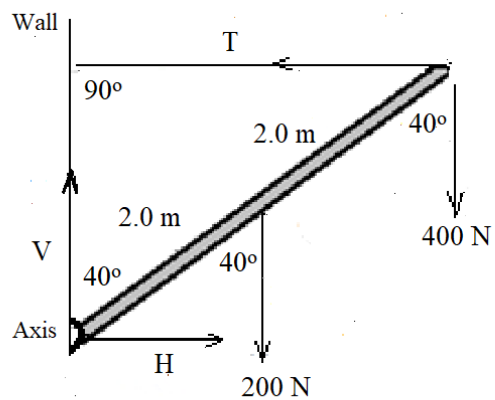
What is the total torque acting on the rod?

continued……



Force (N)	F_{perp} (N)	d (m)	Torque Direction	Sign	$\tau = \pm$ d F_{perp} (m-N)
300	300	3.0	clockwise	+	900
C	C	0	N.A.	N.A.	0
600	600 sin 30^o = 300	6.0	counter-clockwise	-	-1800
900	900	1.5	clockwise	+	1350

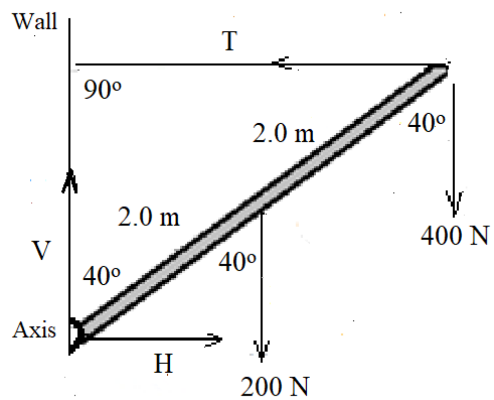
Total Torque: τ = 450 m-N

Example: A 200 N uniform pole, 4.0 meters long, is hinged to a wall. Hanging from the top of the pole is a 400-N object. A horizontal wire from the wall attached to the top of the pole prevents it from rotating. The beam is "in equilibrium," meaning that its linear and angular accelerations are zero. What is the tension (T) in the wire and the vertical (V) and horizontal (H) components of the force exerted by the wall's hinge on the pole? Note: Earth's pull on the uniform pole may be treated as if all of the pull is applied to the center of the pole. Vertical Components $a_y = 0$, so the sum of the y-components of the forces is zero: $V - 200\text{ N} - 400\text{ N} = 0$ $V = 600\text{ N}$ Horizontal Components $a_x = 0$, so the sum of the x-components of the forces is zero: $H - T = 0$ $H = T$	

Once we determine the value of T, we will know what is the value of H. So, there is only one unknown remaining (T). The table below shows that the force values of H and T are still to be determined, but we know from the work above that H = T. So, once we determine T, we will know the value of H.

On the next page, we will determine the value of T using the fact that the sum of the torques is zero.

The pole and its forces are shown again, below, for easy reference.



Force (N)	d (m)	F_{perp} (N)	Torque Direction	$\tau = \pm dF_{perp}$ (m-N)
H	0	N.A.	N.A.	0
V	0	N.A.	N.A.	0
200	2	200 sin 40	Clockwise (+)	2 (200 sin 40)
400	4	400 sin 40	Clockwise (+)	4 (400 sin 40)
T	4	T cos 40	Counter-Clockwise (-)	- 4 (T cos 40)

Rotational Analog of Newton's Second Law

In an earlier chapter we learned that if the total force an object experiences is zero, the object's linear acceleration is zero. In a similar way, if the total torque acting on an object is zero, the *angular* acceleration is zero, and vice-versa.

We now use this law to determine the tension in the wire in the previous example. The pole's angular acceleration is zero, so the sum of the torques is zero:

$$2\,(200 \sin 40) + 4\,(400 \sin 40) - 4\,(T \cos 40) = 0$$

$$T = 419.55 \text{ N}$$

From our earlier work, we now know the value of H:

$$H = 419.55 \text{ N}$$

Rotational Inertia

Symbol: I
Units: kg-m^2

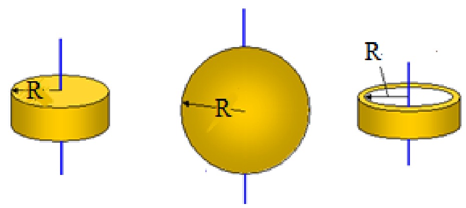
The property of objects called "mass" is a measure of how hard it is to linearly accelerate the object. Another name for "mass" is "linear inertia."

"Rotational inertia" is the analog of linear inertia. The greater the rotational inertia of an object, the harder it is to *angularly* accelerate it.

The rotational inertia of an object depends not only on its mass, but also *where* the mass is relative to the object's axis of rotation:

The greater the percentage of mass that is far from the axis of rotation, the harder it is to angularly accelerate the object, i.e., the greater is its rotational inertia.

The objects below all have same mass M and radius, R, but one of them has the greatest percentage of its mass far from the axis of rotation, and therefore has the greatest rotational inertia.



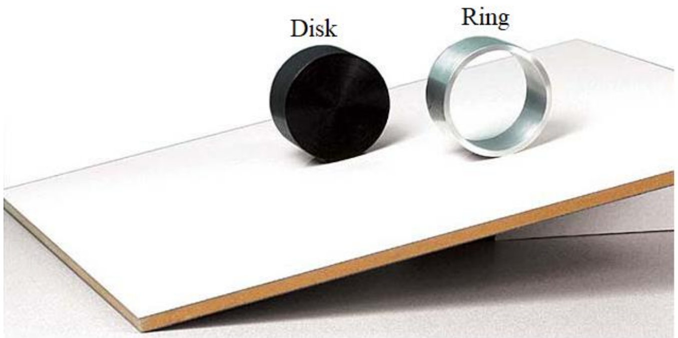
Disk $I = 1/2\ MR^2$ | Sphere $I = 2/5\ MR^2$ | Ring $I = MR^2$

The ring has the largest rotational inertia because it has a greater percentage of its mass far from the axis than do the disk and sphere.

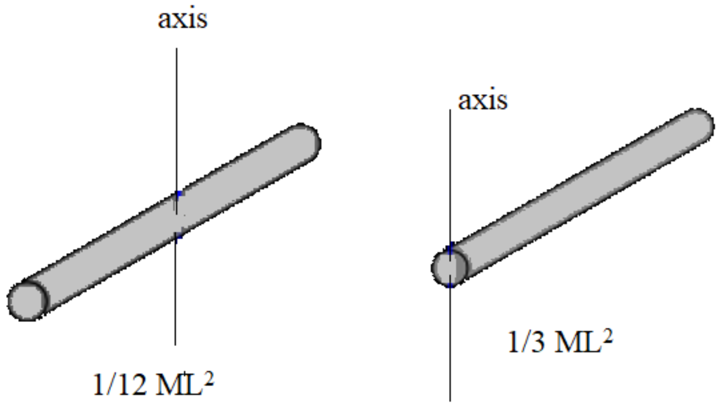
In a three-object race down a ramp by a disk, ring, and sphere, all of equal mass and radius, which object would reach the bottom first?

Ring and Disk Race

The ring and disk below have equal masses and radii. The race to the bottom is won by the object that can more easily be angularly accelerated. Therefore the disk, whose rotational inertia (½ MR^2) is half that of the ring (MR^2), wins the race.



Rotational Inertia of Rods

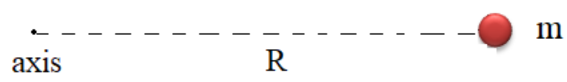


Notice above that the rod with an axis at the end has the greater rotational inertia; that's because a greater percentage of its mass is far from the axis, compared to the rod with an axis through its center.

Rotational Inertia of Point Masses

“Point” masses are objects that have diameters that are negligibly small compared to their distance to the axis of rotation.

The point mass below is a distance R from an axis of rotation.



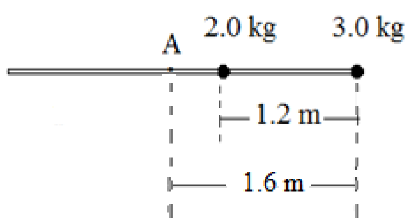
$I = mR^2$

Rotational Inertias are Additive

Example A:

A rod, 3.2 m long, has a mass of 4.0 kg. Point-sized objects are attached to the rod at the points indicated in the figure below.

What is the rotational inertia of this object, relative to an axis through the center, at Point A?



Note that the distance from the 2.0-kg mass to the axis A is 0.4 m.

The total rotational inertia is the sum of the three rotational inertias. Recall that the rotational inertia of a rod relative to an axis at its center is (1/12) ML^2.

$I = (1/12)\,(4.0)(3.2)^2 + (2.0)(0.4)^2 + (3.0)(1.6)^2$
$= 11.4$ kg-m^2

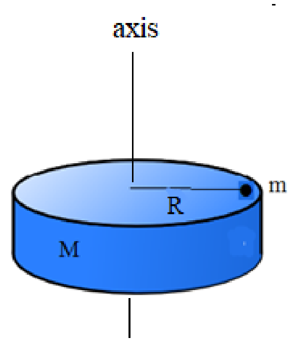
Example B:

A disk of mass M = 5.0 kg has a radius R = 0.40 m. On the rim of the disk rests a point mass m = 0.80 kg.

What is the rotational inertia of this object relative to an axis through the center?

Recall the inertia equations for a disk, and a point mass:

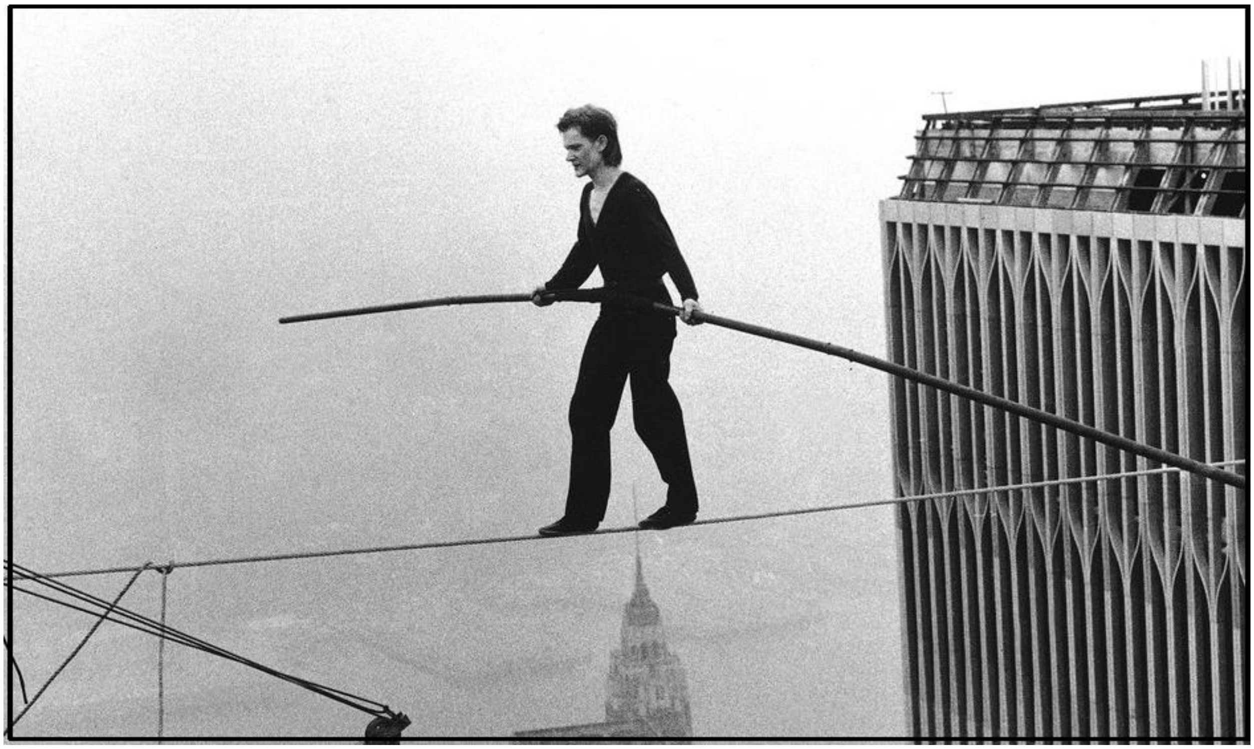
$I = ½\,MR^2$ and $I = mR^2$



$I = ½\,MR^2 + mR^2$
$= ½\,(5.0)\,(0.40)^2 + 0.80\,(0.40)^2$
$= 0.53$ kg-m^2

Additive Inertia in Tightrope Walking

The greater the rotational inertia of an object, the harder it is to angularly accelerate it. The person below increases her rotational inertia by adding the pole's inertia to her own, thereby making it less likely she'll tip over.



North tower of the World Trade Center (1973-2001) is in the background.