Physics 23 Chapter 5

Dr. Joseph F. Alward

Circular Motion

	Video Lecture 1: Introduction to Centripetal Acceleration Video Lecture 2: The Radial Axis Video Lecture 3: Object Orbiting in a Vertical Plane

Velocity is a vector quantity: it has a magnitude, and it has a direction. If either one of these two changes, then the object has an acceleration.

One kind of acceleration you've already seen--the kind that changes speed. Another kind we will discuss in this chapter is the kind that changes the object's direction, but not its speed. This second kind of acceleration that changes direction is called "centripetal acceleration" and it occurs in circular motion.

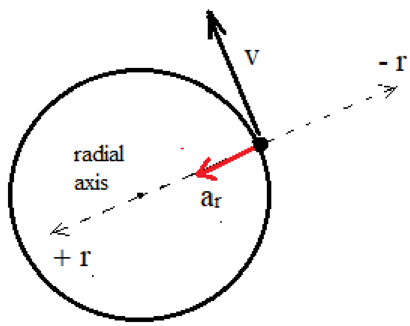
Centripetal accelerations always points toward the center of the circle, in what is called the "radial" direction. Wherever the object is at any moment, the radial direction points from that point to the center of the circle. Thus, in circular motion, there are an infinite number of radial directions, but at any given instant only one of them is relevant.

Symbol: a_r

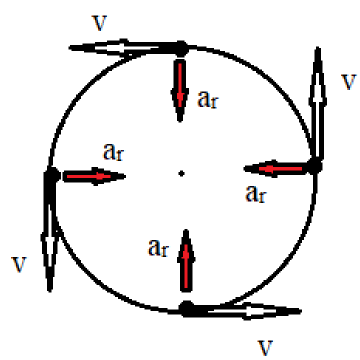
We state without proof the following:

a_r = "centripetal acceleration"
$= v^2/r$

a_r is always positive



$a_r = v^2/r$



Centripetal acceleration always points to the center of the circle, in the positive radial direction (+r direction).

Example A:

An object is traveling at a constant speed of 4.0 m/s in a circular path of radius 2.0 m.

What is its acceleration?

$$a = v^2/r$$

Sometimes we suppress the subscript on the centripetal acceleration symbol "a"--as we did above--if we're sure the reader will know what acceleration we're talking about.

$$\begin{aligned} a &= (4.0\ \text{m/s})^2/2.0\ \text{m} \\ &= 8.0\ \text{m/s}^2 \end{aligned}$$

Example B:

An object is moving in a circular path of radius r = 1.2 m, and is completing 0.40 revolutions per second.

What is the object's centripetal acceleration?

Solution:

In one revolution the object travels a distance equal to the circumference of the circle.

$$\begin{aligned} \text{Circumference} &= 2\pi\,(1.2) \\ &= 7.54\ \text{m} \end{aligned}$$

In one second, the object travel 0.40 revolution, a distance of

0.40 (7.54 m) = 3.02 m

$v = 3.02$ m/s

$$\begin{aligned} a &= (3.02\ \text{m/s})^2/1.2\ \text{m} \\ &= 7.60\ \text{m/s}^2 \end{aligned}$$

Example:

The centripetal acceleration of an object moving at the end of a string is 4.0 m/s^2.

If the string's length (r) is doubled, and the object's speed (v) is tripled, what would be the new centripetal acceleration?

Initially: $a = v^2/r$
$= 4.0$ m/s^2

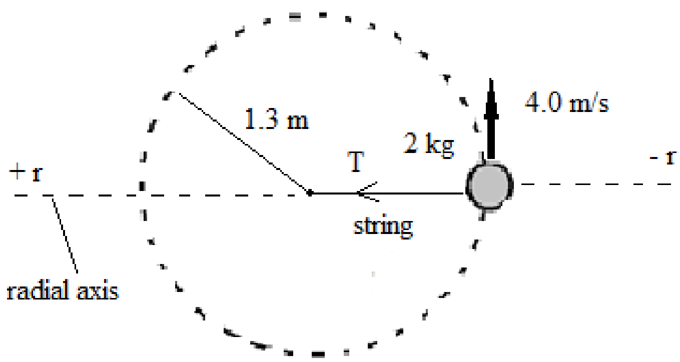
Finally: $a = (3v)^2/(2r)$
$= (9/2)\, v^2/r$
$= (9/2)\,(4.0)$
$= 18$ m/s^2

Example:

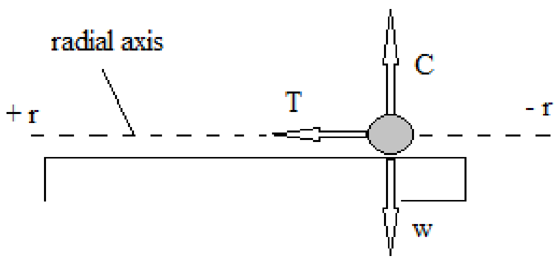
The object in the figure below is moving in a circular path on a frictionless tabletop. Three forces act on the object; one is the tension in the string; another is the object's weight, and the third force is the contact force between the tabletop and the object. What is the tension in the string?

Note that the positive radial direction points to the left, toward the center of the circle.

View from Above



Side View

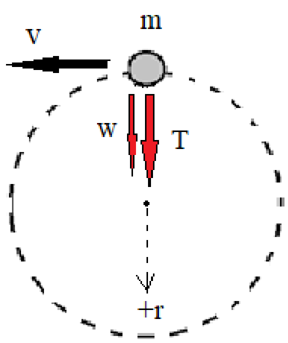


$$F_r = ma_r$$
$$T_r + w_r + C_r = mv^2/r$$
$$T + 0 + 0 = mv^2/r$$
$$T = 2.0\,(4.0)^2/1.3$$
$$= 24.6\text{ N}$$

Example A:

An object of mass $m = 3.0$ kg is moving in a *vertical* plane at 12.0 m/s in a circular path at the end of a string of length 2.4 m.

What is the tension in the string?

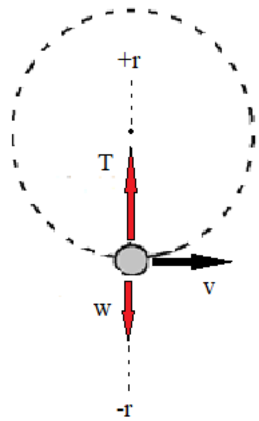


(Positive radial direction points downward, toward the center of the circle.)

$$F_r = ma_r$$
$$T_r + w_r = mv^2/r$$
$$T + mg = mv^2/r$$
$$T = mv^2/r - mg$$
$$= 3.0\,(12.0)^2/2.4 - 3.0\,(9.8)$$
$$= 150.6 \text{ N}$$

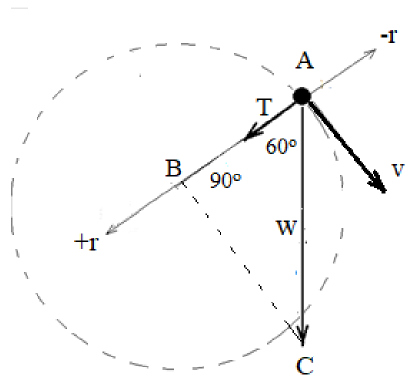
Example B:

What is the tension in the string when the object in Example A is at the bottom of its swing?



(Positive radial direction is upward, toward the center of the circle.)

$$F_r = ma_r$$
$$T_r + w_r = mv^2/r$$
$$T - mg = mv^2/r$$
$$T = mg + mv^2/r$$
$$= 3.0\,(9.8) + 3.0\,(12.0)^2/2.4$$
$$= 209.4 \text{ N}$$

Example:	
In the figure at the right, an object of mass m = 2.0 kg is revolving at the end of the string in a vertical circular path of radius r = 1.2 m at a speed of v = 5.0 m/s. The string makes an angle of 60° relative to the vertical direction. What is the tension T in the string? Note: The weight vector is the hypotenuse of the ABC right triangle: $F_r = ma_r$ $T_r + w_r = mv^2/r$ $T_r + mg \cos 60 = mv^2/r$ $T + (2)(9.8) \cos 60 = (2)(5.0)^2/1.2$ $T = 31.87$ N	

Example A:

A race car travels at speed v around a frictionless circular track of radius r "banked" at an angle θ.

Derive an expression for the required banking angle θ in terms of r and v.

The y-coordinate of the car never changes (it's always y = 0), so $a_y = 0$ and therefore the sum of the y-components of the forces acting on the race car is zero: $F_y = 0$.

Apply Newton's 2nd Law to the y-direction:

$$F_y = 0$$
$$C_y + w_y = 0$$
$$C \cos\theta - mg = 0$$
$$C \cos\theta = mg \quad \text{(Equation 1)}$$

Apply Newton's 2nd Law to the radial direction.

$$F_r = ma_r$$
$$C_r + w_r = ma_r$$
$$C \sin\theta + 0 = mv^2/r$$
$$C \sin\theta = mv^2/r \quad \text{(Equation 2)}$$

Divide Equation 2 by Equation 1:

$$\frac{C \sin\theta = mv^2/r}{C \cos\theta = mg}$$

$$\tan\theta = v^2/gr$$
$$\theta = \mathrm{Tan}^{-1}(v^2/gr)$$

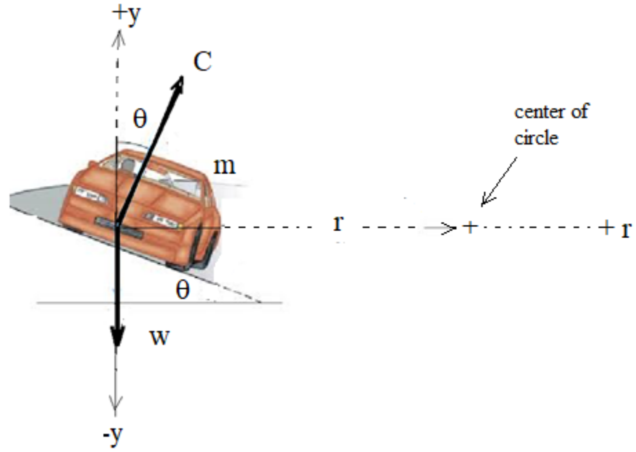
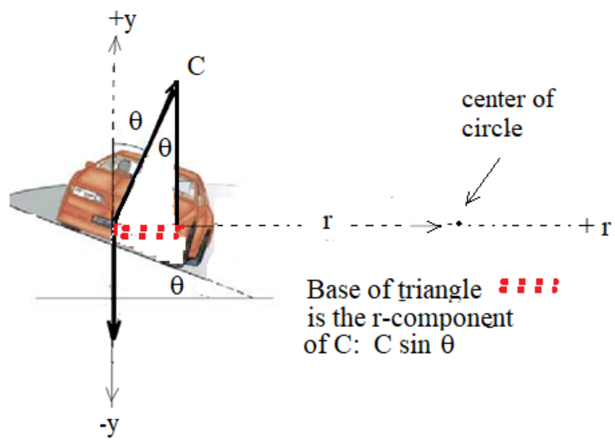


Diagram below explains why the r-component of C is C sin θ.



Example B:

The radius of a circular freeway ramp exit is 160 meters.

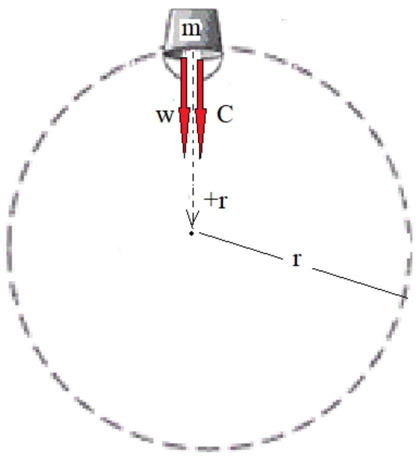
What must be the banking angle to ensure a safe exit at 20 m/s, assuming the worst conditions--icy road and zero friction?

$$\theta = \mathrm{Tan}^{-1}(v^2/gr)$$
$$= 14.3^\circ$$

	Video: Nascar race cars travel around a banked track

	Video: Swinging bucket of water overhead

Example:



The water in a bucket being swung in a vertical circle at the beach is a distance r = 0.60 m from the center of the circle. What is the least speed v the water could have without the water losing contact with the bottom of the bucket, and then falling onto the head of the person swinging the bucket?

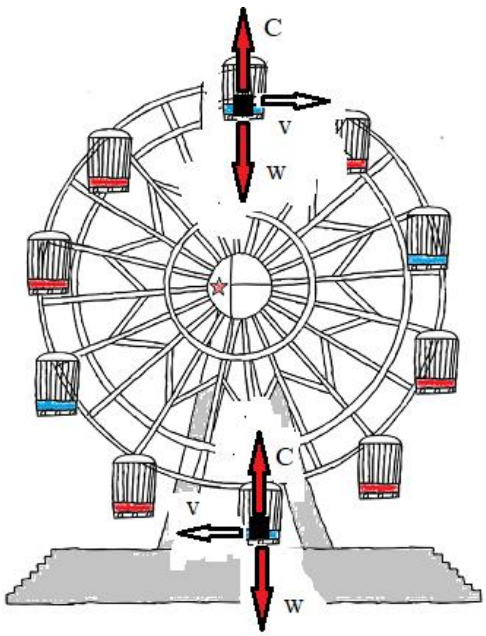
$$ma_r = F_r$$
$$mv^2/r = C_r + w_r$$

Let C_r = 0.0000001 N (virtually zero). Water is still in contact with the bottom of the bucket, but just barely.

$$mv^2/r = 0 + mg$$
$$v = (rg)^{1/2}$$
$$= 2.42 \text{ m/s}$$

Example:

Below is shown a Ferris Wheel*, with passengers in gondolas at the top and bottom. The tangential speed of the passengers is v. The lengths (magnitudes) of the contact force C and weight force w are indicated.



Apply Newton's Second Law to passengers at the top, and at the bottom, of the wheel:

At the top, the positive radial direction points from the passenger downward toward the center of the Ferris wheel

The C vector points upward in the negative radial direction, so $C_r = -C$.

The weight force w points downward in the positive radial direction, so $w_r = + mg$.

$$F_r = ma_r$$
$$C_r + w_r = ma_r$$
$$-C + mg = m(v^2/r)$$

At the bottom, the positive radial direction points from the passenger upward toward the center of the Ferris wheel.

The C vector points upward in the positive radial direction, so $C_r = +C$.

The weight force w points downward in the negative radial direction, so $w_r = - mg$.

$$F_r = ma_r$$
$$C_r + w_r = ma_r$$
$$C - mg = m(v^2/r)$$

*The original **Ferris Wheel** was designed and constructed by George Washington Gale **Ferris**, Jr. as a landmark for the 1893 World's Columbian Exposition in Chicago. With a height of 264 ft. it was the largest attraction when it opened to the public on June 21, 1893.

Satellites in Circular Orbits

<table>
<tr>
<td>

<u>Example</u>:

The figure at the right shows a cannon at the equator firing projectiles. At lower speeds, the projectile strikes the ground, but at a certain minimum speed, the projectile orbits Earth as it returns to its starting point.

(a) At what speed would a cannon have to fire a projectile horizontally in order that it orbit Earth and eventually return to the cannon, ignoring air resistance and obstacles such as trees, buildings and hills?

$G = 6.67 \times 10^{-11}$ N-m^2/kg^2
$M = 5.98 \times 10^{24}$ kg
$R = 6.38 \times 10^6$ m (about 4,000 miles)

$ma_r = F_r$
$mv^2/R = GMm/R^2$

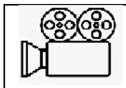
$v = (GM/R)^{1/2}$
$= 7906$ m/s

</td>
<td>

(b) How long would it take the projectile to return to the location from which it was thrown?

Time = Distance / Speed
= Circumference/Speed
$= 2\pi\,(6.38 \times 10^6)/7906$
= 5070 s
= 84.51 minutes
= 1.41 hours

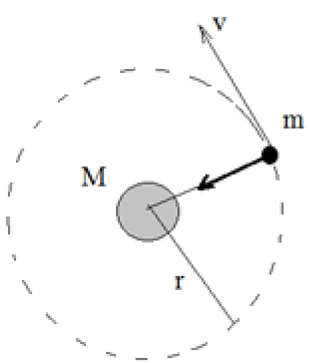
</td>
</tr>
</table>

	Planetary Orbit Animation The orbital distances and relative speeds of the eight planets are shown. This video demonstrates that all of the planets orbit counter-clockwise (as viewed from above the orbital plane, and that the smaller the orbital radius, the greater is the orbital speed.

<u>Example</u>:

Derive the relationship between the radius of a planetary circular orbit and the planet's orbital speed.

Earth's greatest distance from the Sun is about three percent greater than the shortest distance, so Earth's orbit is not perfectly circular. Nevertheless, in what follows, we *assume that Earth's orbit is circular;* this assumption will introduce only a very small error in our calculations.



$$F_r = ma_r$$
$$GMm/r^2 = mv^2/r$$
$$rv^2 = GM$$
$$= \text{constant}$$

The product of the orbital radius and the orbital speed, squared, is the same (equal to GM) for all planets. This constancy is illustrated in the equation below for Planets 1 and 2:

$$r_2v_2^2 = r_1v_1^2$$

Example:

Two planets have circular orbits about a certain star. Planet 1 has an orbital speed of 30 km/s.

Planet 2 is four times farther from the star than Planet 1:

$(r_1/r_2) = ¼$

What is its orbital speed?

$r_2 v_2^2 = r_1 v_1^2$

$v_2 = (r_1 / r_2)^{1/2} \, v_1$

$= (1/4)^{1/2} \, 30$

$= 15$ km/s

Note: It is not always necessary to use standard international (SI) units in problems. In this case, input speed in kilometers per second, rather than meters per second, and take as the result an answer in kilometers per second.

If we had used 30,000 m/s for Planet 1, our calculation would have given us 15,000 m/s as the speed of Planet 2.

Orbital Speed and Orbital Period

Orbital Speed

The equation ($rv^2 = GM$) derived above for circular satellite orbits is valid for circular orbits of planets around any star, including our own star--the sun.

$M = 2.0 \times 10^{30}$ kg (mass of Sun)
$G = 6.67 \times 10^{-11}$ N-m^2/kg^2
Distance Earth to Sun: $r = 1.5 \times 10^{11}$ m

$$rv^2 = GM$$
$$v = (GM/r)^{1/2}$$
$$= (6.67 \times 10^{-11} \times 2.0 \times 10^{30}/1.5 \times 10^{11})^{1/2}$$
$$= 2.98 \times 10^4 \text{ m/s}$$

Orbital Period?

The distance traveled by Earth as it completes one trip around the star is the circumference of the circle:

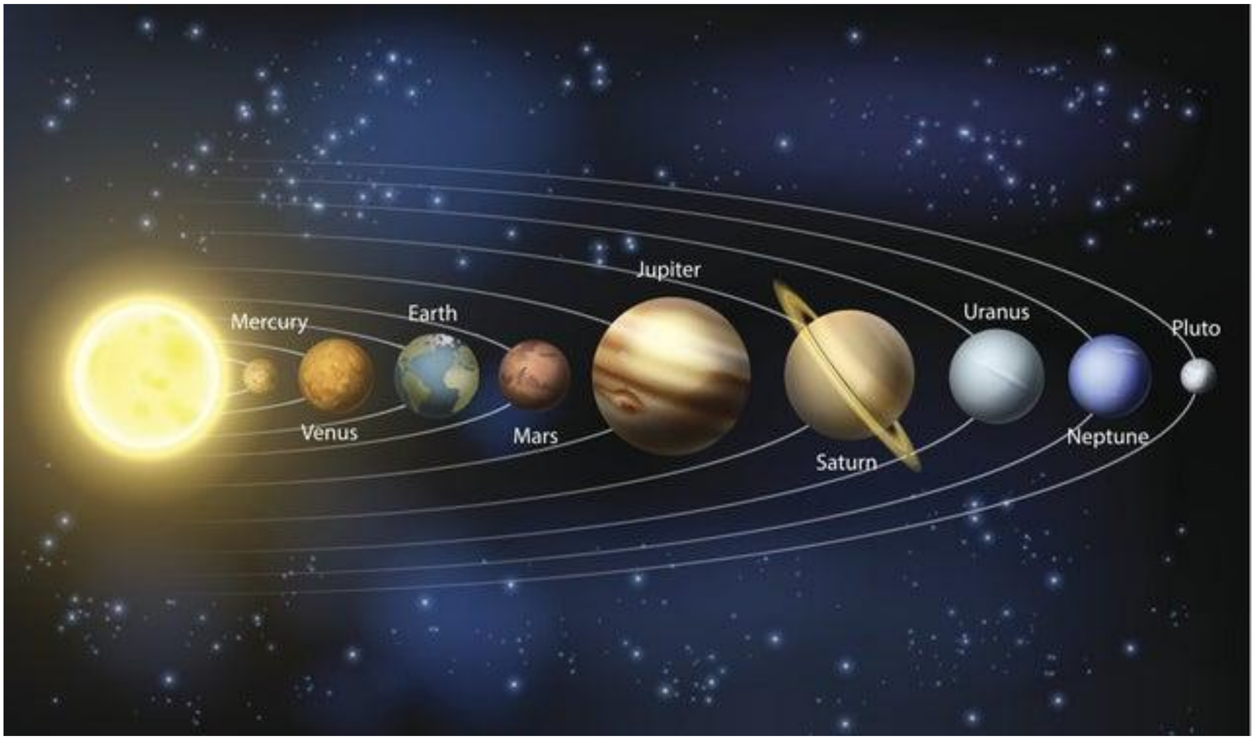
$$\text{Circumference} = 2\pi\,(1.5 \times 10^{11} \text{ m})$$
$$= 9.42 \times 10^{11} \text{ m}$$

The time it takes for Earth to complete one orbit around the sun is symbolized as T:

T = Distance / Speed
= Circumference/Speed

$$T = (9.42 \times 10^{11})/\ (2.98 \times 10^4)$$
$$= 3.16 \times 10^7 \text{ s}$$
$$= 365.61 \text{ days}$$

The actual value is approximately 365.25 days; the difference compared to our calculated value is due to our treating Earth's orbit in our calculation as if it were perfectly circular, which is not exactly true.



<u>Example</u>:

Suppose an alternate universe exists in which the gravitational force law is $F = Gm_1m_2/r^4$, while all other laws of physics are the same as in this universe.

How does the orbital speed v of planets in circular orbits depend on the orbital radius r in the alternate universe?

$$
\begin{aligned}
F &= ma \\
GMm/r^4 &= mv^2/r \\
r^3v^2 &= GM \\
v^2 &= GM/r^3 \\
&= GMr^{-3} \\
v &= (GM)^{1/2}\, r^{-3/2} \\
v &\sim r^{-3/2}
\end{aligned}
$$

Example:

A planet orbiting a star in the imagined alternative universe in the previous example has a speed of 20 km/s.

What is the orbital speed of a planet that is six times farther from the star?

$v_2/v_1 = r_2^{-3/2} / r_1^{-3/2}$
$= r_1^{3/2} / r_2^{3/2}$
$= (r_1/r_2)^{3/2}$
$v_2/20 = (1/6)^{3/2}$
$v_2 = 1.36$ km/s