# Physics 23 Chapter 3 

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## Motion in a Plane



Video Lecture 1: Introduction to Motion in a Plane
Video Lecture 2: Projectile Motion Introduction Part 1
Video Lecture 3: Projectile Motion Introduction Part 2
Video Lecture 4: Projectile Motion Problem 1
Video Lecture 5: Projectile Motion Problem 2

Objects moving on an $x-y$ coordinate plane have $x$-and $y$-components of location, velocity, and acceleration. The equations governing this motion are identical to the ones for linear motion studied in Chapter 2, except that subscripts are used to distinguish between the x - and y -components of the displacement, velocity, and acceleration.

$$
\begin{array}{ll}
(\mathrm{x}, \mathrm{y}) & =\text { final coordinates of the object } \\
\mathrm{v}_{\mathrm{o}} & =\text { initial speed (magnitude of initial velocity) } \\
\mathrm{v}_{\mathrm{ox}} & =\mathrm{x} \text {-component of initial velocity } \mathrm{v}_{\mathrm{o}} \\
\mathrm{v}_{\mathrm{oy}} & =\mathrm{y} \text {-component of initial velocity } \mathrm{v}_{\mathrm{o}} \\
\mathrm{v}_{\mathrm{x}} & =\text { final } \mathrm{x} \text {-component of velocity } \\
\mathrm{v}_{\mathrm{y}} & =\text { final y-component of velocity } \\
\mathrm{a} & =\text { acceleration (assumed constant) } \\
\mathrm{a}_{\mathrm{x}} & =\mathrm{x} \text {-component of acceleration } \mathrm{a} \\
\mathrm{a}_{\mathrm{y}} & =y \text {-component of acceleration } \mathrm{a}
\end{array}
$$

Not all of the horizontal or vertical equations studied in Chapter 2 will be used in this chapter. The ones that will be used are shown below:

| $v_{x}=v_{o x}+a_{x} t$ | $v_{y}=v_{o y}+a_{y} t$ |
| :--- | :--- |
| $x=v_{o x} t+1 / 2 a_{x} t^{2}$ | $y=v_{o y} t+1 / 2 a_{y} t^{2}$ |

## Example:

The data in the figure at the right provide the initial $x$ - and $y$-components of the velocities and accelerations of a rocket traveling in a vertical plane. The rocket is initially at the origin $(0,0)$.
(a) What will be the $x$ - and $y$-coordinates of the rocket three seconds later?

$$
\begin{aligned}
\mathrm{x} & =\mathrm{v}_{\text {oxt }}+1 / 2 \mathrm{a}_{\mathrm{x}} \mathrm{t}^{2} \\
& =180(3)+1 / 2(60) 3^{2} \\
& =810 \mathrm{~m} \\
\mathrm{y} & =v_{\text {oy }} \mathrm{t}+1 / 2 \mathrm{a}_{\mathrm{y}} \mathrm{t}^{2} \\
& =100(3)+1 / 2(20) 3^{2} \\
& =390 \mathrm{~m}
\end{aligned}
$$



(b) How far from its initial location will the rocket be at that time?

The distance in question is the hypotenuse of the right triangle whose base width is 810 m , and whose leg height is 390 m :
$\mathrm{d}=\left(\mathrm{x}^{2}+\mathrm{y}^{2}\right)^{1 / 2}$
$=\left(810^{2}+390^{2}\right)^{1 / 2}$
$=899 \mathrm{~m}$

## Projectile Motion

"Projectiles" are objects that are struck, thrown, launched, or fired forward and upward or downward, traveling in a vertical plane only under the influence of gravity. Struck golf balls, shot arrows, and fired bullets execute projectile motion.
A rocket, for example, does not execute projectile motion until all of its fuel has been exhausted and it's moving on its own--"freely"-- only under the influence of Earth's pull (gravity), without any propulsion device (rocket engine) aiding its motion.
In projectile motion, the object is not only moving horizontally, but also vertically. Therefore, there are two velocities-- horizontal, and vertical, as well as two kinds of directions of travel-horizontal, and vertical. The path the projectile travels is called its "trajectory."

## Fundamental Fact Relating to Projectile Motion:

Vertical and horizontal motion occur independently of each other. The entertaining video below describes this independency.


Click here to watch the video.

In all of the projectile motion problems we will study below, it is assumed that the initial x - and $y$-coordinates of the projectile are zero, i.e., the projectile is initially at the coordinates $(0,0)$.

## Vertical Component of Projectile Motion

At all points along the projectile's trajectory Earth is constantly pulling down on the projectile. On its way up, the projectile loses vertical velocity at exactly the same rate that it would if the ball were moving only upward, subtracting $9.8 \mathrm{~m} / \mathrm{s}$ from its upward velocity with each passing second, until zero vertical velocity is reached and maximum height is reached. Likewise, on its way back down, the projectile's velocity becomes progressively more negative by the amount $9.8 \mathrm{~m} / \mathrm{s}$ with each second of downward travel.

The figure below shows how the vertical velocity changes in the case where the initial vertical velocity is $49 \mathrm{~m} / \mathrm{s}$.


On its way down from its peak, the projectile's vertical velocity, starting at zero, becomes progressively more and more negative. With each passing second while moving downward, 9.8 $\mathrm{m} / \mathrm{s}$ is subtracted from the vertical velocity. All the while, the horizontal velocity (not shown) doesn't change.

Note the symmetry of the motion in the figure above: the speed (absolute value of the velocity) upward at a certain elevation is the same as the speed downward at the same elevation. For example, two seconds before landing on the ground the velocity is $-29.4 \mathrm{~m} / \mathrm{s}$, and two seconds after leaving the ground earlier the velocity is $+29.4 \mathrm{~m} / \mathrm{s}$

## Horizontal Velocity in Projectile Motion

Unlike the vertical velocities that constantly are changing during projectile motion, horizontal velocities in projectile motion never change, assuming we can ignore the retardation effects of air resistance. In the example trajectory shown below, the ball's horizontal velocity throughout its motion is the same as it was initially: $12 \mathrm{~m} / \mathrm{s}$


## Total Flight Time

Calculate the total flight time for projectile motion of a golf ball traveling ground-to-ground over level ground.

$$
\mathrm{v}_{\mathrm{oy}}=\text { Initial vertical velocity }
$$

The golf ball begins its journey on the ground, at $\mathrm{y}=0$, and ends its journey on the ground, at $\mathrm{y}=0$.


$$
y=v_{\text {oy }} t-1 / 2 g t^{2}
$$

Let $\mathrm{y}=0$ and solve for t :

$$
\mathrm{t}=2\left(\mathrm{v}_{\mathrm{oy}} / \mathrm{g}\right)
$$

This is the total time it takes for the object to rise upward from the ground $(y=0)$, then fall back down to the ground $(y=0)$.

The equations below are the main ones needed to solve problems involving projectiles.

| $v_{x}=v_{o x}$ |
| :--- |
| $x=v_{o x} t$ |
| $y=v_{o y} t-1 / 2 g t^{2}$ |
| $v_{y}=v_{\text {oy }}-g t$ |

## Fall Time and Rise Time

For projectiles traveling ground to ground over level ground, the fall time is the same as the rise time, as the proof below will demonstrate.

We showed earlier with the golfer that the total time the ball takes to fly from ground to ground is

$$
\mathrm{t}=2\left(\mathrm{v}_{\mathrm{oy}} / \mathrm{g}\right)
$$

The rise time is the time it takes the ball's vertical velocity to be reduced to zero, at which point the ball has reached maximum height, and it then begins its fall to ground.
$\mathrm{v}_{\mathrm{y}}=\mathrm{v}_{\mathrm{oy}}-\mathrm{gt}$
$0=v_{\text {oy }}-g t$
$\mathrm{t}=\mathrm{v}_{\mathrm{oy}} / \mathrm{g}$ (This is the rise time.)
Fall Time $=$ Total Time - Rise Time

$$
\begin{aligned}
& =2\left(\mathrm{v}_{\mathrm{oy}} / \mathrm{g}\right)-\mathrm{v}_{\mathrm{yo}} / \mathrm{g} \\
& =\mathrm{v}_{\mathrm{oy}} / \mathrm{g}(\text { This is the fall time. })
\end{aligned}
$$

Thus, the fall time is the same as the rise time.

## Rise and Fall Times in Projectile Motion

The diagram below a shows a football's "trajectory" over level ground. The ball is initially on the ground when kicked. It travels along the path indicated and lands on the ground. The initial horizontal velocity of the ball is $20.0 \mathrm{~m} / \mathrm{s}$, and the initial vertical velocity is $19.6 \mathrm{~m} / \mathrm{s}$.


| Example A: | Solution: |
| :---: | :---: |
| A bullet is fired from a pistol at a certain elevation above the ground, and leaves the pistol with an initial vertical velocity of $98 \mathrm{~m} / \mathrm{s}$, and an initial horizontal velocity of $120 \mathrm{~m} / \mathrm{s}$. | The rise time is the time it takes for the initial vertical velocity to be reduced to zero: $\begin{aligned} & \mathrm{v}_{\mathrm{y}}=\mathrm{v}_{\mathrm{oy}}-\mathrm{gt} \\ & 0=98-9.8 \mathrm{t} \\ & \mathrm{t}=10 \mathrm{~s} \end{aligned}$ |
| $98 \mathrm{~m} / \mathrm{s}$ | Rise time: 10 s |
|  | Flight time $=$ Twice Rise Time: |
|  | $\begin{aligned} x & =v_{\text {ox }} t-1 / 2 a_{x} t^{2} \\ & =120(20)-0 \\ & =2400 \mathrm{~m} \end{aligned}$ |
| How far horizontally does the bullet travel before returning to the same elevation from which it was fired? | The horizontal distance a projectile travels between two points at the same elevation is called the "range," R. In this example, $R=2400 \mathrm{~m}$. |


| Example B: | Solution: |
| :---: | :---: |
| A football kicked from the | $\mathrm{x}=\mathrm{v}_{\text {ox }} \mathrm{t}-1 / 2 \mathrm{ax} \mathrm{t}^{2}$ |
| ground with an initial horizontal | $75=25 \mathrm{t}+0$ |
| velocity of $25 \mathrm{~m} / \mathrm{s}$ lands on the | $\mathrm{t}=3.0 \mathrm{~s}$ |
| grass 75 m away. | = Flight Time |
| What was the ball's initial vertical velocity? | $\begin{aligned} \text { Rise Time } & =1 / 2(\text { Flight Time }) \\ & =1 / 2(3.0 \mathrm{~s}) \\ & =1.5 \mathrm{~s} \end{aligned}$ |
|  | After 1.5 seconds of flight, the football's vertical velocity is zero: |
|  | $\begin{aligned} \mathrm{v}_{\mathrm{y}} & =\mathrm{v}_{\mathrm{oy}}-\mathrm{gt} \\ 0 & =\mathrm{v}_{\mathrm{oy}}-9.8(1.5) \end{aligned}$ |
|  | $\mathrm{v}_{\text {oy }}=14.7 \mathrm{~m} / \mathrm{s}$ |

## Velocity Components

If the launch angle $\theta$ and the initial speed $\mathrm{v}_{\mathrm{o}}$ are known, one may determine the initial vertical and horizontal velocities of a projectile.


## Rise Time of Projectiles

The rise time of a projectile is the time it takes the vertical velocity to be reduced to zero.
$\mathrm{v}_{\mathrm{y}}=\mathrm{v}_{\mathrm{oy}}-\mathrm{gt}$
$0=\left(v_{o} \sin \theta\right)-g t$
$\mathrm{t}=\left(\mathrm{v}_{\mathrm{o}} \sin \theta\right) / \mathrm{g}$

## Example:

Projectiles leave a cannon at $90 \mathrm{~m} / \mathrm{s}$.
(a) At what angle should the projectiles be fired in order that their flight time over level ground be 6.0 seconds?

As shown earlier, rise time is

$$
\mathrm{t}=\left(\mathrm{v}_{\mathrm{o}} \sin \theta\right) / \mathrm{g}
$$

Furthermore, the rise time is half the flight time,
$3.0=90 \sin \theta / 9.8$
$\theta=19.07^{\circ}$
(b) How far horizontally would the projectiles travel?

$$
\begin{aligned}
\mathrm{x} & =\mathrm{v}_{\mathrm{ox}} \mathrm{t}-1 / 2 \mathrm{ax}_{\mathrm{x}} \mathrm{t}^{2} \\
& =\left(\mathrm{v}_{\mathrm{o}} \cos \theta\right) \mathrm{t}-0 \\
& =90 \cos (19.07) 6.0 \\
& =510.4 \mathrm{~m}
\end{aligned}
$$

| Example: |  |
| :---: | :---: |
| A golf ball struck over level ground by a golf club leaves the clubface at an angle of $22^{\circ}$ relative to the ground with an initial speed of $54 \mathrm{~m} / \mathrm{s}$. | (c) Find the flight time by a second method. $\begin{aligned} \mathrm{v}_{\mathrm{oy}} & =54 \sin (22) \\ & =20.229 \mathrm{~m} / \mathrm{s} \end{aligned}$ |
| (a) What is its rise time? | $\begin{aligned} & \mathrm{y}=\mathrm{v}_{\mathrm{oy}} \mathrm{t}-1 / 2 \mathrm{gt}^{2} \\ & 0=20.229 \mathrm{t}-4.9 \mathrm{t}^{2} \end{aligned}$ |
| $\mathrm{v}_{\mathrm{y}}=\mathrm{v}_{\text {oy }}-\mathrm{gt}$ |  |
| $0=\left(v_{0} \sin \theta\right)-\mathrm{gt}$ | $\mathrm{t}=4.128 \mathrm{~s}$ |
| $\begin{aligned} \mathrm{t} & =\left(\mathrm{v}_{\mathrm{o}} \sin \theta\right) / \mathrm{g} \\ & =(54) \sin 22 / 9.8 \\ & =2.064 \mathrm{~s} \end{aligned}$ | (d) How far horizontally did it travel? |
| (b) What is its flight time? | $\begin{aligned} \mathrm{x} & =\mathrm{v}_{\text {ox }} \mathrm{t}-1 / 2 \mathrm{axx}^{t^{2}} \\ & =(54 \cos 22) 4.128-0 \\ & =206.69 \mathrm{~m} \end{aligned}$ |
| The flight time over level ground is twice the rise time: |  |
| $2(2.064)=4.128 \mathrm{~s}$ |  |

## Projectile Range versus Firing Angle

Recall, projectiles fired over level ground have
a flight time that is twice the rise time:

| t | $=$ Flight Time |
| ---: | :--- |
|  | $=2($ Rise Time $)$ |
|  | $=2\left(\mathrm{v}_{\mathrm{o}} \sin \theta\right) / \mathrm{g}$ |
| R | $=\mathrm{v}_{\mathrm{ox}} \mathrm{t}$ |
|  | $=\left(\mathrm{v}_{\mathrm{o}} \cos \theta\right) \mathrm{t}$ |
|  | $\left.=\left(\mathrm{v}_{\mathrm{o}} \cos \theta\right)\left[2\left(\mathrm{v}_{\mathrm{o}} \sin \theta\right) / \mathrm{g}\right)\right]$ |
|  | $=\left(\mathrm{v}_{\mathrm{o}}^{2} / \mathrm{g}\right) 2 \sin \theta \cos \theta$ |

An important trigonometric identity: $\sin (2 \theta)=2 \sin \theta \cos \theta$
$\mathrm{R}=\left(\mathrm{V}_{\mathrm{o}}{ }^{2} / \mathrm{g}\right) \sin (2 \theta)$
R is largest when $\sin (2 \theta)$ is largest, which occurs when $2 \theta=90^{\circ}$, or $\theta=45^{\circ}$.

## The Complementarity Rule

Verify the following statement:
For projectiles fired over level ground, the range at one firing angle is the same as the range at the complement of the angle. Recall that two angles are the "complement" of each other if their sum is $90^{\circ}$.

Recall from the previous page, $R=\left(\mathrm{v}_{\mathrm{o}}{ }^{2} / \mathrm{g}\right) \sin 2 \theta$ :
Verify for the 75-15 complementary angles

| $75^{\circ}$ | $15^{\circ}$ |
| :--- | :--- |
| $\sin \left(150^{\circ}\right)$ | $\sin \left(30^{\circ}\right)$ |
| 0.500 | 0.500 |

Verify for the 10-80 complementary angles

| $10^{\circ}$ | $80^{\circ}$ |
| :--- | :--- |
| $\sin \left(20^{\circ}\right)$ | $\sin \left(160^{\circ}\right)$ |
| 0.342 | 0.342 |

Verify for the 30-60 complementary angles

| $30^{\circ}$ | $60^{\circ}$ |
| :--- | :--- |
| $\sin \left(60^{\circ}\right)$ | $\sin \left(120^{\circ}\right)$ |
| 0.866 | 0.866 |



Note, for example, a projectile fired at $75^{\circ}$ has the same range as a projectile fired at $15^{\circ}$, and that one fired at $30^{\circ}$ lands at the same spot as one fired at $60^{\circ}$.

## Example A:

An athlete throws a baseball 40 meters. How far would the ball had traveled if it had been thrown with a speed $20 \%$ greater?

We saw earlier that the range of a projectile fired over level ground is given by the equation

$$
\mathrm{R}=\left(\mathrm{v}_{\mathrm{o}}{ }^{2} / \mathrm{g}\right) \sin (2 \theta)
$$

If $\mathrm{v}_{\mathrm{o}}$ increases to 1.20 times the earlier value, then $\mathrm{v}_{0}{ }^{2}$ becomes $1.20^{2}=1.44$ times the earlier value. Therefore, the new range will be 1.44 times the earlier one:
$\mathrm{R}=1.44$ (40)
$=56.6 \mathrm{~m}$

| Example B: |  |  |
| :---: | :---: | :---: |
| A bomber is flying level at $115 \mathrm{~m} / \mathrm{s}$, at an altitude of 1000 m . |  |  |
| How far horizontally from the target should the bomb be released? | 1000 m | ! |
| $\mathrm{y}=\mathrm{v}_{\text {oy }} \mathrm{t}-1 / 2 \mathrm{gt}{ }^{2}$ |  | 1 |
| $-1000=0-1 / 2(9.8) \mathrm{t}^{2}$ |  | 1 |
| $\mathrm{t}=14.29 \mathrm{~s}$ |  | 1 |
| $\begin{aligned} \mathrm{x} & =\mathrm{v}_{\text {ox }} \mathrm{t} \\ & =115(14.29) \\ & =1643 \mathrm{~m} \end{aligned}$ |  | $x=\text { ? }$ |

## Example:

A bomber plane traveling at $150 \mathrm{~m} / \mathrm{s}$ at an altitude of 200 m , diving toward the ground along a direction $15^{\circ}$ below the horizontal, drops a bomb. How far horizontally does the bomb travel before striking the target?


Calculate travel time:

$$
\begin{aligned}
\mathrm{y} & =\mathrm{v}_{\text {oy }} \mathrm{t}-1 / 2 \mathrm{gt}^{2} \\
-200 & =\left(-150 \sin 15^{\circ}\right) \mathrm{t}-4.9 \mathrm{t}^{2} \\
\mathrm{t} & =3.56 \mathrm{~s} \\
\mathrm{x} & =\mathrm{v}_{\text {ox }} \mathrm{t} \\
& =(150 \cos 15) 3.56 \\
& =516 \mathrm{~m}
\end{aligned}
$$

To hit a particular target on the ground, the bomb should be released when the horizontal distance between the bomber and the target is 516 m .

How would the solution above change if the bomber were climbing at $15^{\circ}$, instead of diving?

Determine travel time:
$-200=(150 \sin 15) t-4.9 \mathrm{t}^{2}$
$\mathrm{t}=11.48 \mathrm{~s}$
$\mathrm{x}=(150 \cos 15) 11.48$
$=1663 \mathrm{~m}$
Note that the falling bomb and the bomber travel at the same horizontal velocity, so the bomber is always directly above the bomb. To avoid getting caught in the bomb blast, the plane should change its direction of travel soon after dropping the bomb.

## Example:

At what angle $\theta$ must the projectile below be fired in order to strike the cliff at the point indicated?


Time to the cliff:

$$
\begin{aligned}
\mathrm{x} & =\mathrm{v}_{\mathrm{ox}} \mathrm{t}+1 / 2 \mathrm{axt}^{2} \\
170 & =(200 \cos \theta) \mathrm{t}+0 \\
\mathrm{t} & =0.85 / \cos \theta \\
\mathrm{y} & =\mathrm{v}_{\mathrm{oy}} \mathrm{t}-1 / 2 \mathrm{gt}^{2} \\
20 & =[200 \sin \theta](0.85 / \cos \theta)-4.9(0.85 / \cos \theta)^{2} \\
\theta & =7.91^{\circ}
\end{aligned}
$$

