Physics 23 Chapter 3

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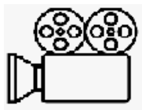
Motion in a Plane



Video Lecture 1: Introduction to Motion in a Plane
Video Lecture 2: Projectile Motion Introduction Part 1
Video Lecture 3: Projectile Motion Introduction Part 2
Video Lecture 4: Projectile Motion Problem 1
Video Lecture 5: Projectile Motion Problem 2

Objects moving on an x-y coordinate plane have x-and y-components of location, velocity, and acceleration. The equations governing this motion are identical to the ones for linear motion studied in Chapter 2, except that subscripts are used to distinguish between the x- and y-components of the displacement, velocity, and acceleration:

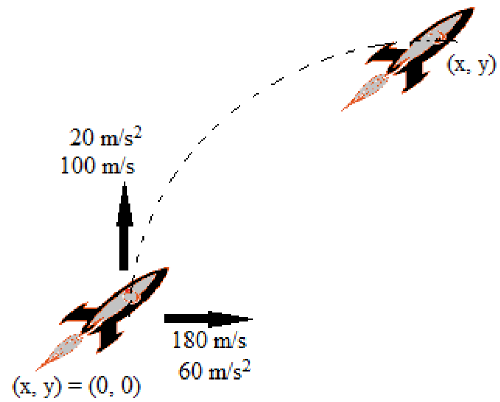
(x, y) = final coordinates of the object
v_o = initial speed (magnitude of initial velocity)
v_{ox} = x-component of initial velocity
v_{oy} = y-component of initial velocity
v_x = final x-component of velocity
v_y = final y-component of velocity
a = acceleration (assumed constant)
a_x = x-component of (acceleration a
a_y = y-component of acceleration a

The equations in this chapter that will be needed to solve the problems discussed in this chapter are the ones below.

$v_x = v_{ox} + a_x t$	$v_y = v_{oy} + a_y t$
$x = v_{ox} t + ½ a_x t^2$	$y = v_{oy} t + ½ a_y t^2$

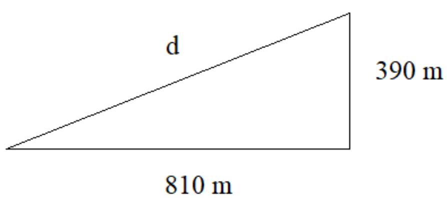
Example:

The data in the figure at the right provide the initial x- and y-components of the velocities and accelerations of a rocket traveling in a vertical plane. The rocket is initially at the origin (0, 0).



(a) What will be the x- and y-coordinates of the rocket three seconds later?

$x = v_{ox}t + ½\, a_x t^2$
$= 180\ (3) + ½\ (60)\ 3^2$
$= 810$ m
$y = v_{oy}t + ½\, a_y t^2$
$= 100\ (3) + ½\ (20)\ 3^2$
$= 390$ m



(b) How far from its initial location will the rocket be at that time?

The distance in question is the hypotenuse of the right triangle whose base width is x = 810 m, and whose leg height is y = 390 m, as shown in the figure at the left.

$d = (x^2 + y^2)^{1/2}$
$= (810^2 + 390^2)^{1/2}$
$= 899$ m

Projectile Motion

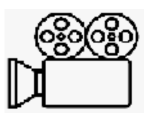
“Projectiles” are objects that are struck, thrown, launched, or fired forward and upward or downward, traveling in a vertical plane only under the influence of gravity. Struck golf balls, shot arrows, and fired bullets execute projectile motion, for example.

A rocket, for example, does not execute projectile motion until all of its fuel has been exhausted and it’s moving on its own--“freely”-- only under the influence of Earth’s pull (gravity), without any propulsion device (rocket engine) aiding its motion. Such “missiles” are described as “ballistic,” as in inter-continental ballistic missiles (ICBMs)

In projectile motion, the object is not only moving horizontally, but also vertically. Therefore, there are two velocities-- horizontal, and vertical, as well as two kinds of directions of travel-- horizontal, and vertical.   The path the projectile travels is called its “trajectory.”

<u>Fundamental Fact Relating to Projectile Motion:</u>

Vertical and horizontal motion occur *independently* of each other. The entertaining video below describes this independency.



Click <u>here</u> to watch the video.

The equations below are the main ones needed to solve problems involving projectiles.

$v_x = v_{ox}$
$x = v_{ox}\, t$
$y = v_{oy}\, t - ½ gt^2$
$v_y = v_{oy} - gt$

In all of the projectile motion problems we will study below, it is assumed that the projectile is initially at the coordinates (0, 0).

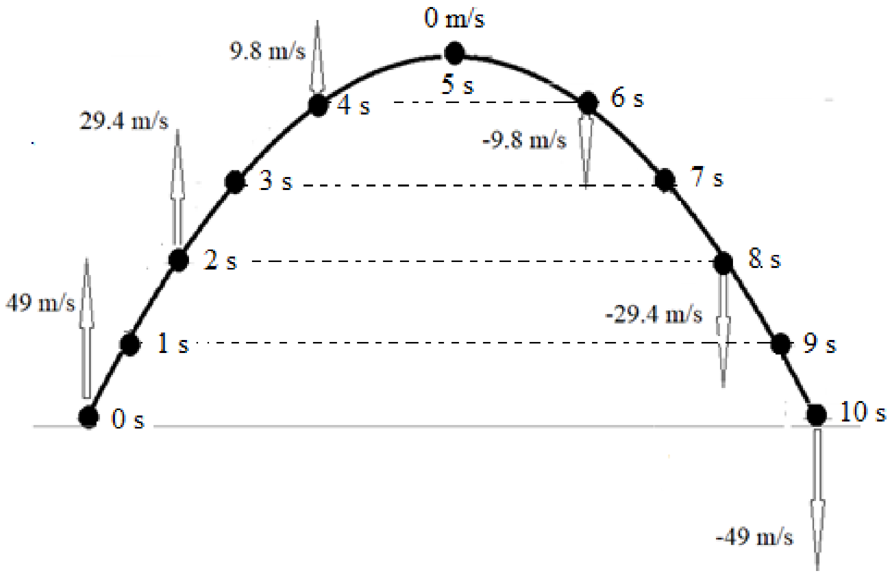
Vertical Component of Projectile Motion

At all points along the projectile's trajectory Earth is constantly pulling downward on the projectile. On its way up, the projectile loses vertical velocity at exactly the same rate that it would if the ball were moving only upward, subtracting 9.8 m/s from its vertical velocity with each passing second, until zero vertical velocity and maximum height is reached. Likewise, on its way back down, the projectile's velocity becomes progressively more negative by the amount 9.8 m/s with each second of downward travel.

The figure below shows how the vertical velocity changes in the case where the initial vertical velocity is 49 m/s.

$$v_y = v_{oy} - gt$$

$$v_y = 49\text{ m/s} - (9.8\text{ m/s}^2)\ t$$

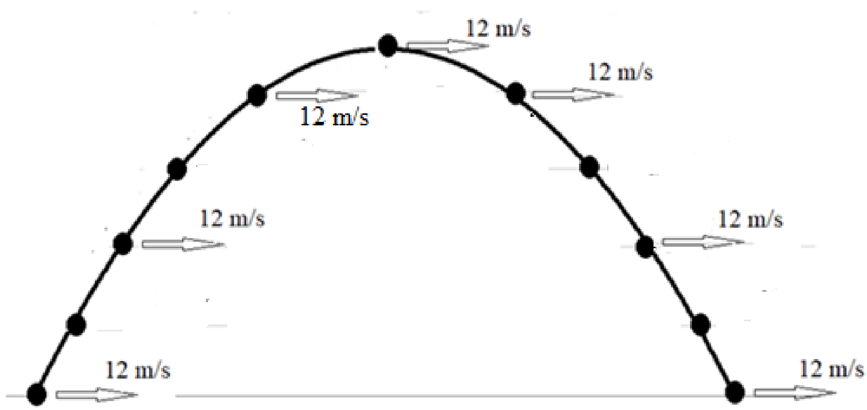


On its way down from its peak, the projectile's vertical velocity, starting at zero, becomes progressively more and more negative. With each passing second while moving downward, 9.8 m/s is subtracted from the vertical velocity. All the while, the horizontal velocity (not shown) doesn't change.

Note the symmetry of the motion in the figure above: the speed (absolute value of the velocity) upward at a certain elevation is the same as the speed downward at the same elevation. For example, two seconds before landing on the ground the velocity is -29.4 m/s, and two seconds *after* leaving the ground earlier the velocity is +29.4 m/s

Horizontal Velocity in Projectile Motion

Unlike the vertical velocities that constantly are changing during projectile motion, horizontal velocities in projectile motion never change, assuming we can ignore the retardation effects of air resistance. In the example trajectory shown below, the ball's horizontal velocity throughout its motion is the same as it was initially: 12 m/s



Total Flight Time

Calculate the total flight time for projectile motion of a golf ball traveling ground-to-ground over level ground.

v_{oy} = Initial vertical velocity

The golf ball begins its journey on the ground, at y = 0, and ends its journey on the ground, again, at y = 0. Calculate the amount of time of necessary to complete this trip. Call this time, "the flight time."

$y = v_{oy}\, t - ½\, gt^2$

Let y = 0 and solve for t:

$t = 2\, (v_{oy}/g)$ (flight time)

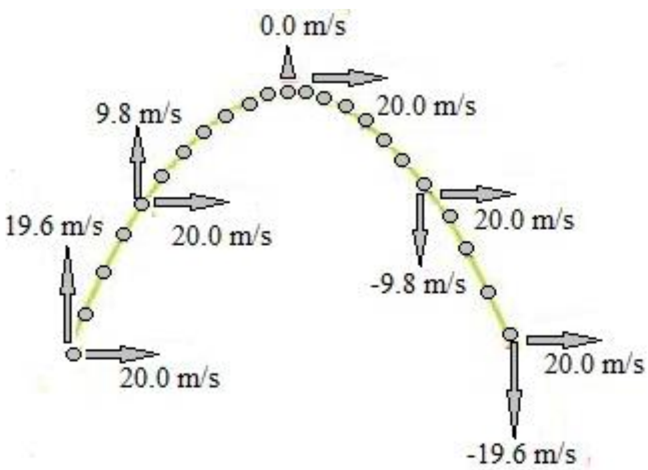
Fall Time and Rise Time

For projectiles traveling ground to ground over level ground, the time it takes the projectile to rise to its maximum heigh (the "rise time) is that same as the time it takes the projectile to fall from its maximum height, down to the ground (the "fall time"). We will prove this fact, below.

Rise and Fall Times in Projectile Motion

The diagram below a shows a football's "trajectory" over level ground. The ball is initially on the ground when kicked. It travels along the path indicated and lands on the ground. The initial horizontal velocity of the ball is 20.0 m/s, and the initial vertical velocity is 19.6 m/s.

<u>Example</u>:



Note: In projectile motion, horizontal velocity doesn't change. In this case, the unchanging horizontal velocity is 20.0 m/s.

(a) What is the rise time?

The rise time is the time it takes for Earth's gravitational pull to reduce the vertical velocity to zero, at which time the football has reached its maximum height.

$v_y = v_{oy} - gt$
$0 = 19.6 - 9.8t$
$t = 2.0$ s

(b) What is the maximum height?

Maximum height is reached when the vertical velocity is reduced to zero: $v_y = 0$

$y = ½ (v_{oy} + v_y) t$
$= ½ (19.6 + 0) 2.0$
$= 19.6$ m

………………………………..

(c) What is the total flight time, ground to ground?

$y = v_{oy} t - ½ gt^2$
$0 = 19.6t - 4.9 t^2$
$t = 4.0$ seconds

………………………………..

(d) What is the fall time?

Fall time = Total Time - Rise Time
= 4.0 - 2.0
= 2.0 s

The <u>fall</u> time for projectiles traveling between locations of the same elevation is the same as the <u>rise</u> time. Thus, the <u>total</u> flight time is double the rise time.

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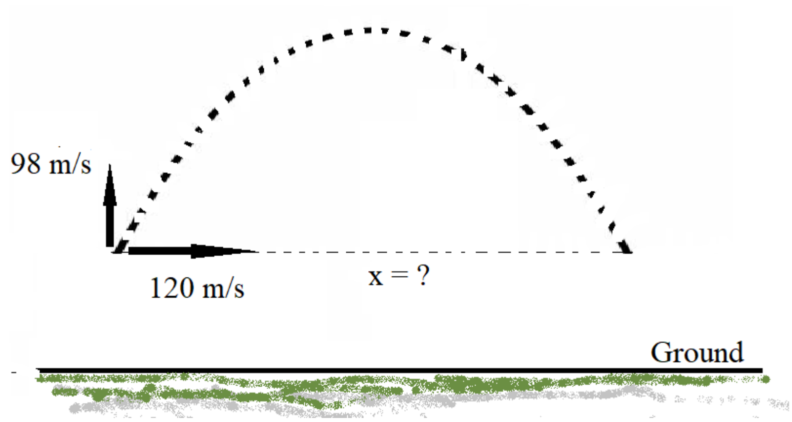
<u>Example A</u>:

A bullet is fired from a pistol at a certain elevation above the ground, and leaves the pistol with an initial vertical velocity of 98 m/s, and an initial horizontal velocity of 120 m/s.



How far horizontally does the bullet travel before returning to the same elevation from which it was fired?
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<u>Solution</u>:

The rise time is the time it takes for the initial vertical velocity to be reduced to zero:

$v_y = v_{oy} - gt$
$0 = 98 - 9.8\ t$
$t = 10\ s$

Rise time: 10 s

Flight time = Twice Rise Time:
= 20 s

$x = v_{ox}\ t - \frac{1}{2}\ a_x t^2$
$= 120\ (20) - 0$
$= 2400\ m$

The horizontal distance a projectile travels between two points at the same elevation is called the "range," R. In this example, R = 2400 m.
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<u>Example B</u>:

A football kicked from the ground with an initial horizontal velocity of 25 m/s lands on the grass 75 m away.

What was the ball's initial vertical velocity?
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<u>Solution</u>:

$x = v_{ox}\ t - \frac{1}{2}\ a_x t^2$
$75 = 25\ t + 0$
$t = 3.0\ s$
= Flight Time

Rise Time = ½ (Flight Time)
= ½ (3.0 s)
= 1.5 s
After 1.5 seconds of flight, the football's vertical velocity is zero:

$v_y = v_{oy} - gt$
$0 = v_{oy} - 9.8(1.5)$
$v_{oy} = 14.7\ m/s$
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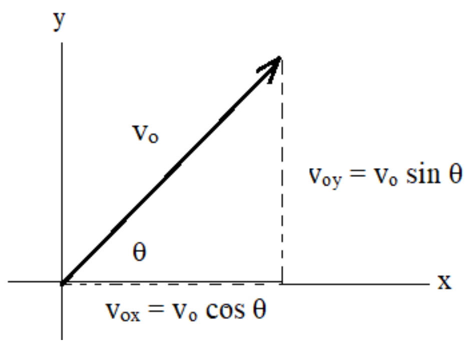
Velocity Components

If the launch angle θ and the initial speed v_o are known, one may determine the initial vertical and horizontal velocities of a projectile, as shown below:



Rise Time of Projectiles

The rise time of a projectile is the time it takes the vertical velocity to be reduced to zero.

$v_y = v_{oy} - gt$
$0 = (v_o \sin\theta) - gt$
$t = (v_o \sin\theta) / g$

Note that this time—the rise time—is the same as the fall time, which means that the flight time is twice the fall time (or, twice the rise time).

Example:

Projectiles leave a cannon at 90 m/s.

(a) At what angle should the projectiles be fired in order that their flight time over level ground be 6.0 seconds?

As shown earlier, rise time is

$t = (v_o \sin\theta) / g$

Furthermore, the rise time is half the flight time,

$3.0 = 90 \sin\theta / 9.8$
$\theta = 19.07^o$

(b) How far horizontally would the projectiles travel? (Recall: $a_x = 0$ for projectiles)

$x = v_{ox}\, t - ½\, a_x t^2$
$= (v_o \cos\theta)\, t - 0$
$= 90 \cos(19.07)\ 6.0$
$= 510.4$ m

Example:	(c) Find the flight time by a second method.
A golf ball struck over level ground by a golf club leaves the clubface at an angle of 22° relative to the ground with an initial speed of 54 m/s.	$v_{oy} = 54 \sin (22)$ $= 20.229$ m/s
(a) What is its rise time?	$y = v_{oy}t - ½ gt^2$ $0 = 20.229\, t - 4.9\, t^2$
$v_y = v_{oy} - gt$ $0 = (v_o \sin \theta) - gt$ $t = (v_o \sin \theta) /g$ $= (54) \sin 22 /9.8$ $= 2.064$ s …………………………………………	$t = 4.128$ s …………………………………………
(b) What is its flight time?	(d) How far horizontally did it travel?
The flight time over level ground is twice the rise time: 2 (2.064) = 4.128 s	$x = v_{ox}\, t - ½\, a_x t^2$ $= (54 \cos 22)\, 4.128 - 0$ $= 206.69$ m

Projectile Range versus Firing Angle

Recall, projectiles fired over level ground have a flight time that is twice the rise time:

t = Flight Time
 = 2 (Rise Time)
 = $2 (v_o \sin \theta)/g$

$R = v_{ox}\, t$
 $= (v_o \cos \theta)\, t$
 $= (v_o \cos \theta)\, [2 (v_o \sin \theta)/g)]$
 $= (v_o^2/g)\, 2 \sin \theta \cos \theta$

An important trigonometric identity:
$\sin (2\theta) = 2 \sin \theta \cos \theta$

$R = (v_o^2/g) \sin (2\theta)$

R is largest when $\sin (2\theta)$ is largest, which occurs when $2\theta = 90^o$, or $\theta = 45^o$.

The Complementarity Rule

Verify the following statement:

For projectiles fired over level ground, the range at one firing angle is the same as the range at the complement of the angle. Recall that two angles are the "complement" of each other if their sum is 90^o.

Recall from the previous page, $R = (v_o^2/g) \sin 2\theta$:

Verify for the 75-15 complementary angles

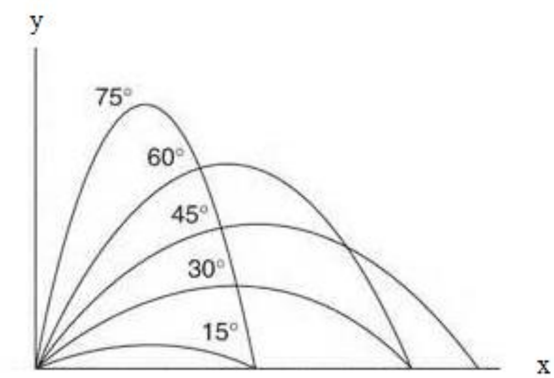
75^o	15^o
$\sin (150^o)$	$\sin (30^o)$
0.500	0.500

Verify for the 10-80 complementary angles

10^o	80^o
$\sin (20^o)$	$\sin (160^o)$
0.342	0.342

Verify for the 30-60 complementary angles

30^o	60^o
$\sin (60^o)$	$\sin (120^o)$
0.866	0.866



Note, for example, a projectile fired at 75^o has the same range as a projectile fired at 15^o, and that one fired at 30^o lands at the same spot as one fired at 60^o.

Example A:

An athlete throws a baseball 40 meters. How far would the ball had traveled if it had been thrown with a speed 20% greater?

We saw earlier that the range of a projectile fired over level ground is given by the equation

$$R = (v_o^2/g) \sin(2\theta)$$

If v_o increases to 1.20 times the earlier value, then v_o^2 becomes $1.20^2 = 1.44$ times the earlier value. Therefore, the new range will be 1.44 times the earlier one:

$$R = 1.44\ (40)$$
$$= 56.6 \text{ m}$$

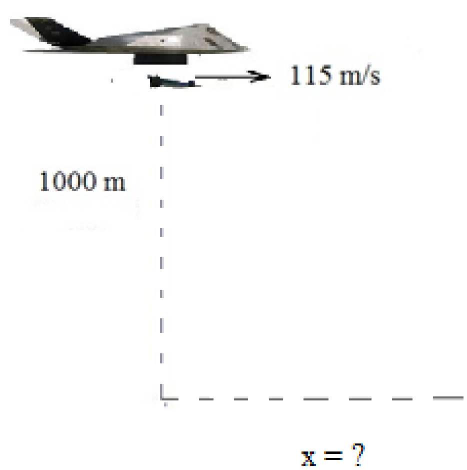
Example B:

A bomber is flying level at 115 m/s, at an altitude of 1000 m.

How far horizontally from the target should the bomb be released?

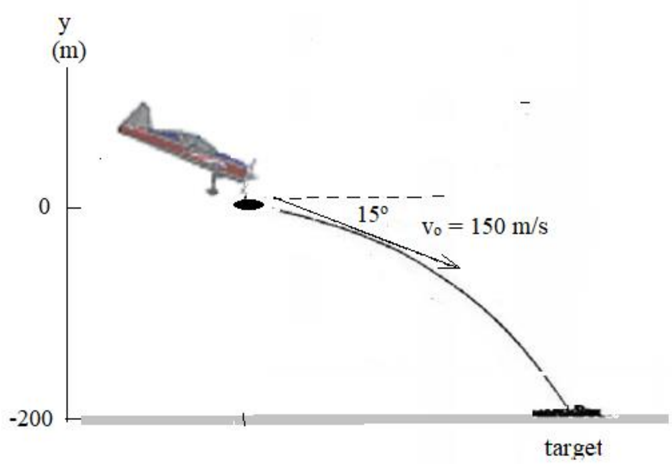
$$y = v_{oy}t - \tfrac{1}{2} gt^2$$
$$-1000 = 0 - \tfrac{1}{2}(9.8)t^2$$
$$t = 14.29 \text{ s}$$

$$x = v_{ox}\, t$$
$$= 115\ (14.29)$$
$$= 1643 \text{ m}$$



Example:

A bomber plane traveling at 150 m/s at an altitude of 200 m, diving toward the ground along a direction 15° below the horizontal, drops a bomb. How far horizontally does the bomb travel before striking the target?



Calculate travel time:

$y = v_{oy}\, t - \frac{1}{2} g t^2$

$-200 = (-150 \sin 15^\circ)\, t - 4.9\, t^2$

$t = 3.56$ s

$x = v_{ox}\, t$

$= (150 \cos 15)\, 3.56$

$= 516$ m

To hit a particular target on the ground, the bomb should be released when the horizontal distance between the bomber and the target is 516 m.

How would the solution above change if the bomber were *climbing* at 15°, instead of diving?

Determine travel time:

$-200 = (150 \sin 15)\, t - 4.9\, t^2$

$t = 11.48$ s

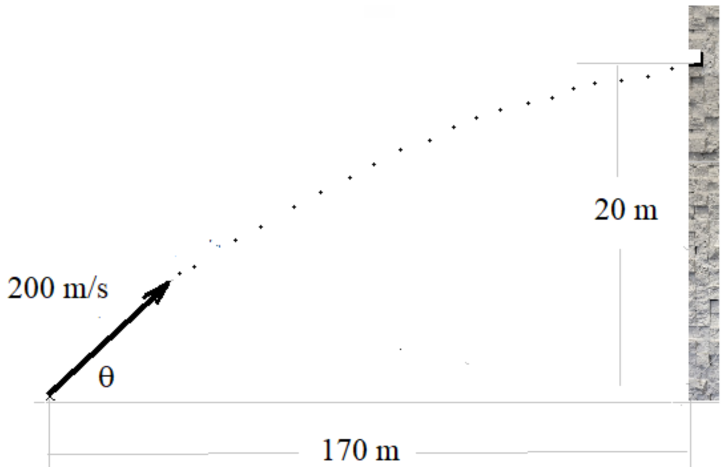
$x = (150 \cos 15)\, 11.48$

$= 1663$ m

Note that the falling bomb and the bomber travel at the same horizontal velocity, so the bomber is always directly above the bomb. To avoid getting caught in the bomb blast, the plane should change its direction of travel soon after dropping the bomb.

Example:

At what angle θ must the projectile below be fired in order to strike the cliff at the point indicated?



Time to the cliff:

$$x = v_{ox}t + \tfrac{1}{2} a_x t^2$$
$$170 = (200 \cos \theta)\, t + 0$$
$$t = 0.85/\cos \theta$$

$$y = v_{oy}t - \tfrac{1}{2} g t^2$$
$$20 = [200 \sin \theta\,]\,(0.85/\cos \theta) - 4.9\,(0.85/\cos \theta)^2$$

$$\theta = 7.91^{o}$$